3/30/2010 36-402/608 ADA-II H. Seltman

Handout #18: Mixed Models, part 3

1. Formal model (review)

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\mathbf{b} + \boldsymbol{\epsilon} \qquad \mathbf{b} \sim N(\mathbf{0}, \mathbf{G}) \qquad \boldsymbol{\epsilon} \sim N(\mathbf{0}, \mathbf{R})$$

$$E(\mathbf{y}) = \mathbf{X}\boldsymbol{\beta} \qquad \mathrm{Var}\,(\mathbf{y}) = \mathbf{Z}\mathbf{G}\mathbf{Z}' + \mathbf{R}$$

2. Random intercept only model:

$$\mathbf{Z} = \begin{pmatrix} 1 & 0 & \cdots \\ \vdots & \vdots & \vdots \\ 1 & 0 & \ddots \\ 0 & 1 & \cdots \\ \vdots & \vdots & \vdots \\ 0 & 1 & \cdots \\ 0 & 0 & \ddots \end{pmatrix}$$

$$\mathbf{G} = \begin{pmatrix} \tau^2 & 0 & \cdots \\ 0 & \tau^2 & \cdots \\ 0 & 0 & \ddots \end{pmatrix}$$

$$\text{With, e.g., } n_1 = 3,\ n_2 = 2, \quad \mathbf{Z}\mathbf{Z}' = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$\mathbf{Z}\mathbf{G}\mathbf{Z}' + \mathbf{R} = \mathbf{Z}\mathbf{G}\mathbf{Z}' + \boldsymbol{\sigma^2}\mathbf{I} = \begin{pmatrix} \sigma^2+\tau^2 & \tau^2 & \tau^2 & 0 & 0 & 0 & \cdots \\ \tau^2 & \sigma^2+\tau^2 & \tau^2 & 0 & 0 & 0 & \cdots \\ \tau^2 & \tau^2 & \sigma^2+\tau^2 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \sigma^2+\tau^2 & \tau^2 & 0 & \cdots \\ 0 & 0 & 0 & \tau^2 & \sigma^2+\tau^2 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \sigma^2+\tau^2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$\Rightarrow \mathrm{Var}\,(y_1) = \mathrm{Var}\,(y_3) = \mathrm{Var}\,(y_4) = \sigma^2 + \tau^2$$

$$\Rightarrow \mathrm{Cor}\,(y_1, y_4) = 0, \quad \mathrm{Cor}\,(y_1, y_3) = \frac{\tau^2}{\sqrt{\sigma^2+\tau^2}\sqrt{\sigma^2+\tau^2}} = \frac{\tau^2}{\sigma^2+\tau^2}$$

3. Dyadic Problems
   (a) "Subjects" are groups of two.
   (b) Examples: married couples (same or opposite sex), friends, work partners, etc.
   (c) Between-dyads predictor variables never vary within the dyad, e.g., couple's treatment assignment, household income, years together.
   (d) Within-dyads predictor variables differ for the members of the dyan, but have the same average for each dyad, e.g., gender (when dyads are the same for gender mix), higher-income-friend, more senior work partner, etc.
   (e) Mixed predictor variables don't meet the above criteria, e.g., age, IQ, and income of each subject.
   (f) Coding of binary within-dyad explanatory variables is commonly 1 vs. -1 to improve interpretability of the intercept and interactions.
   (g) Data setup is two rows per dyad, with a dyad id variable, "actor" outcome, "actor" mixed predictors, "actor" within dyad predictors, between-dyad predictors, and "partner" mixed predictors.
   (h) A mixed model accounts for unmeasured dyad-level explanatory variables (in addition to the usual person level residual variance).
   (i) Extend to larger groups by having "actor" variables vs. "partner average" variables on each line.
4. Breakout and Discussion