Let $J < n$ be the largest subspace not equal to $\mathbb{R}^n$. Let

$$A = \left\{ f : \ ||f - \Pi_j f|| > \epsilon_{2j}, \ j = 1, \ldots, J \right\}. \tag{1}$$

Then $F^*(f) = \{f\}$ for all $f \in A$.

As before we conclude that $\mathbb{P}_f\{\mathcal{R}_j^c\} \leq \alpha_j$ and the power is at least $1 - \alpha_j - \gamma$. To bound the power of such a test, we have the following result.

THEOREM 0.1. *Let δ be the solution to*

$$\frac{n^2 \epsilon_J^4}{n - J} + n^2 \sum_{s=1}^{j} (\epsilon_{s-1}^2 - \epsilon_s^2)^2 = \log(1 + 4\delta^2).$$

Then

$$\beta \equiv \inf_{\phi_\alpha \in \Phi_\alpha} \sup_{f \in A} \mathbb{P}_f\{\phi_\alpha = 0\} \geq 1 - \alpha - \delta$$

where

$$\Phi_\alpha = \left\{ \phi_\alpha : \ \sup_{P \in \mathcal{F}_j} \mathbb{P}_f\{\phi_\alpha = 0\} \leq \alpha \right\}.$$

PROOF. We find a $P_0 \in \mathcal{F}_j$ and a measure μ supported on A such that $||P_0 - P_\mu|| \leq 2\delta$. It then follows that $\beta \geq 1 - \alpha - \delta$.

Let $\psi_1, \psi_2, \ldots, \psi_n$ be an orthonormal basis and let

$$f = \sum_{s=1}^{J} \lambda_s E_s \psi_s + \lambda \sum_{s=J+1}^{n} E_s \psi_s$$

where $(E_s : \ s = 1, \ldots, n)$ are independent Rademacher random variables, that is, $\mathbb{P}\{E_s = 1\} = \mathbb{P}\{E_s = -1\} = 1/2$. Write $\lambda_s = \lambda$ for $s > J$. Now, for each j, $\Pi_j f = \sum_{s=1}^{j} \lambda_s E_s \psi_s$ and hence $||f - \Pi_j f||^2 = \sum_{s=j+1}^{n} \lambda_s^2$. Let $\lambda^2 = (n - J)\epsilon_{2J}^2/n$ and $\lambda_s^2 = n(\epsilon_{s-1}^2 - \epsilon_s^2)$ for $s \leq J$. Then $||f - \Pi_j f|| = \epsilon_{2j}$ for each j and and hence $f \in A$ for each choice of the Rademachers.

Let $Q = \mathsf{E}(P_E)$ where P_E is the distribution under $f = \sum_{s=1}^{n} \lambda_s E_s \psi_s$ and the mean is with repect to the Rademachers. Choose $f_0 \in \mathcal{F}_j$ and let P_0 be the corresponding distribution. As in Baraud, we use the bound

$$||Q - P_0|| \leq \sqrt{\mathsf{E}_0 \left(\frac{dQ}{dP_0}(Y) \right)^2 - 1}.$$

We take $f_0 = (0, \ldots, 0)$ and so

$$\begin{aligned} \left(\frac{dQ}{dP_0}(Y) \right) &= \mathsf{E}_E \left(\exp\left\{ -\frac{1}{2} \sum_{s=1}^{n} \lambda_s^2 + \sum_{s=1}^{n} \lambda_s E_s Y_s \right\} \right) \\ &= e^{-\sum_{s=1}^{n} \lambda_s^2/2} \prod_{s=1}^{n} \cosh(\lambda_s Y_s). \end{aligned}$$

Since $\mathsf{E}_0 \cosh^2(\lambda_j Y_j) = e^{\lambda_j^2} \cosh(\lambda_j^2)$ and $\cosh(x) \leq e^{x^2/2}$ we have

$$\mathsf{E}_0 \left(\frac{dQ}{dP_0}(Y) \right)^2 = \prod_{s=1}^{n} \cosh(\lambda_s^2) \leq e^{\sum_{s=1}^{n} \lambda_s^4/2}. \quad (2)$$

Hence,

$$\beta \geq 1 - \alpha - \frac{1}{2}\sqrt{e^{\sum_{s=1}^{n} \lambda_s^4/2} - 1}.$$

Now note that

$$\sum_{s=1}^{n} \lambda_s^4 \leq \log(1 + 4\delta^2)$$

so the result follows. □

Note: In the single subspace case this reduces to $\epsilon = C(n-J)^{1/4} n^{-1/2}$ which agrees with Baraud.