Nonparametric Inference and the Dark Energy Equation of State

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Road Map

1. Preliminaries
2. Dark Energy
3. Inference for the Dark Energy Equation of State
4. Fisher Follies

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Preliminaries

- The Expanding Universe

  Scale factor $a(t)$ indicates relative expansion of the universe.
  ($a(t_0) = 1$ where t_0 is current age of universe.)

  Redshift z is an observable shift in the wavelength of light from a distant object that is induced by the expansion of the universe.

$$1 + z = \frac{\lambda_{\text{obs}}}{\lambda_{\text{emit}}} = \frac{a(t_{\text{obs}})}{a(t_{\text{emit}})}.$$

  Hubble parameter $H(t) = \frac{\dot{a}(t)}{a(t)}$. ($H_0 = H(t_0)$ is the Hubble "constant".)

- The Distance-Redshift Relation

  The relationship between objects' distances and redshifts contains fundamental information about the Universe's geometry.

  Hubble's Law, $z = H_0 d$, is reasonably accurate for small distances d.

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An Accelerating Universe

- Accelerating Expansion (Reiss et al. 2004, Perlmutter et al. 2004)

  Type Ia supernovae can serve as a "standard candle".

  Observations of many supernovae reveal that the expansion of the universe is *accelerating*.

  This conclusion is supported by other, independent, measurements, including the Cosmic Microwave Background (Spergel et al. 2003) and large-scale structure (Verde et al. 2002).

- "Missing" Matter

  Studies of galaxies and galaxy clusters suggest that the density of matter (dark and light) is $\Omega_M \approx 0.3$ times the critical density ($\rho_{\mathrm{crit}} = 3H^2/8\pi G$).

  The CMB data from WMAP are consistent (so far) with Gaussian primordial density perturbations like those predicted by inflationary models.

  This seems to imply the existence of another field ρ_{DE} with density $\Omega_{\mathrm{DE}} \approx 0.7$ times the critical density.

  But is that true?

What's Going On?

- One satisfying explanation would be that ρ_{DE} corresponds to the quantum vacuum energy.
- Einstein's "mistake," Vacuum Energy, and the Cosmological Constant Problem.
- Possible explanations:
  - Mistaken assumptions, models, or data analysis.

    (Is the universe really accelerating? Is the mismatch between vacuum energies so large?)
  - A failure of General Relativity.

    (Does GR hold over cosmological scales?)
  - Environmental (Anthropic) Selection.

    (Are there domains with different cosmological constants?)
  - (Dynamical) Dark Energy

    (Does the cosmological constant vary over time?)

(See Carrol 2003 for a nice discussion of these hypotheses and the arguments for/against them.)

Dark Energy (cont'd)

- Dark Energy is a smoothly-distributed energy density that dominates the universe ($\sim$ 74% versus $\sim$ 4% for baryonic matter) and provides a negative pressure acting in opposition to gravity.
- Fundamental questions:
  - What is the nature of dark energy?
  - Is it constant or dynamical?
  - How do we measure dark energy?
  - What can we infer about dark energy from these data?
  - What are the implications for cosmological and particle-physics models?

Dark Energy (cont'd)

- What does the acceleration imply about dark energy?

  Let $\rho = \rho_{\rm matter} + \rho_{\rm radiation} + \rho_{\rm DE} + \cdots$ be the total energy density in the universe.

  Friedmann equation:

$$H^2(t) = \left(\frac{\dot{a}(t)}{a(t)}\right)^2 = \frac{8\pi G}{3}\rho - \frac{\kappa}{a^2(t)}$$

  or equivalently,

$$\dot{a}^2 = \frac{8\pi G}{3} a^2 \rho - \kappa.$$

  Acceleration implies that $a^2\rho$ must increase.

  Neither matter ($\rho_{\rm matter} \propto a^{-3}$) nor radiation ($\rho_{\rm radiation} \propto a^{-4}$) can do this.
  A cosmological constant ($\rho_{\rm DE} \propto a^0$) could.

  Note that $\Omega_{\rm M}/\Omega_{\rm DE} \propto a^{-3}$. This gives rise to the Coincidence Problem.

Quantifying Dark Energy: Equation of State

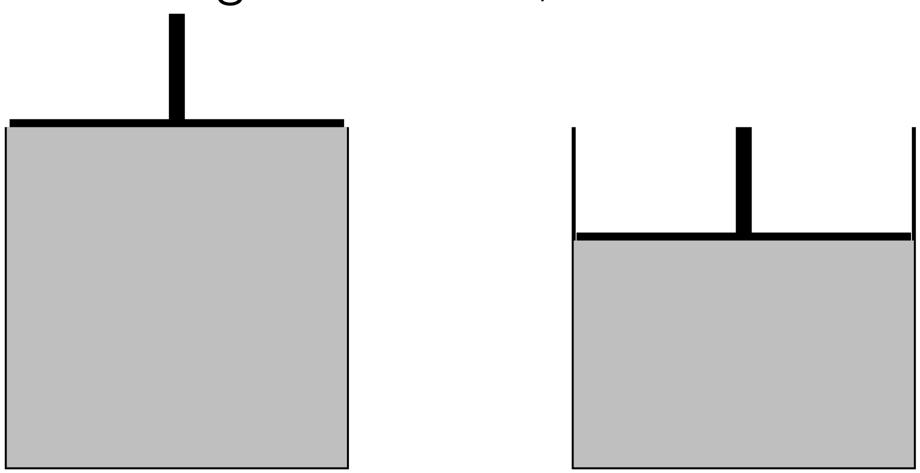
We can attempt to make inferences about ρ directly.

Alternatively, we can look at the equation of state (cf. ideal gas law).

Let p_{DE} and ρ_{DE} be the pressure and energy density of dark energy, then the equation of state w relates these by

$$p_{\mathrm{DE}} = w\rho_{\mathrm{DE}}.$$

For a cosmological constant, $w = -1$.



$$\text{Work} = -p_{\mathrm{DE}}\Delta V$$
$$\Delta\text{Energy} = \rho_{\mathrm{DE}}\Delta V$$
$$\implies p_{\mathrm{DE}} = -\rho_{\mathrm{DE}}$$

The Equation of State (cont'd)

In the special case where $w = w_0$ is constant, then $p_{\rm DE} = w_0 \rho_{\rm DE}$.

Conservation of energy implies:

$$\rho_{\rm DE} \propto a^{-3(1+w_0)}.$$

If $w_0 = -1$, the dark energy density stays constant with time. If $w_0 > -1$, it decreases. And If $w_0 < -1$, it grows.

Cosmologists often restrict the possible energy-momentum tensors with various "energy conditions." A commonly used such condition requires $w \geq -1$, and most cosmological models follow suit.

But it is possible for this condition to be violated, at least over certain time scales. We thus have very little information about the structure of w.

Aside: from the Friedmann equations

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(1 + 3w)\rho,$$

so the Universe will accelerate if $w < -1/3$, and $a(t) \propto t^{\frac{2}{3}(1+w)}$.

Measuring Dark Energy

- Type Ia Supernovae
- Abundances of galaxy clusters (mass function of dark matter halos).

  How this abundance evolves gives information about a and the distance-redshift relation.
- Baryon acoustic oscillations

  Standing wave pattern 140 Mpc, gives standard ruler.
- Integrated Sachs-Wolfe Effect
- Weak Lensing

Measuring Dark Energy: Type Ia Supernovae

Type Ia Supernovae offer standard candles (really standardizable candles) that can help us probe the distance-redshift relation.

- All Type Ia SNe have the same mechanism and thus similar peak brightness

  There remains about 40% scatter in peak brightness among nearby SNe.

- There is a strong correlation between peak brightness and the time it takes to decrease in brightness

  Dimmer SNe decay more rapidly, brighter SNe decay more slowly. This allows one-parameter fit that reduces scatter significantly.

- Potential systematic errors are thought to be small

  Main sources: i. intrinsic differences between Type Ia SNe at low and high redshift and ii. extinction from intergalactic dust (reddening)

- Corroboration with other independent measurements

  Especially CMB and large-scale structure.

Type Ia Supernovae (cont'd)

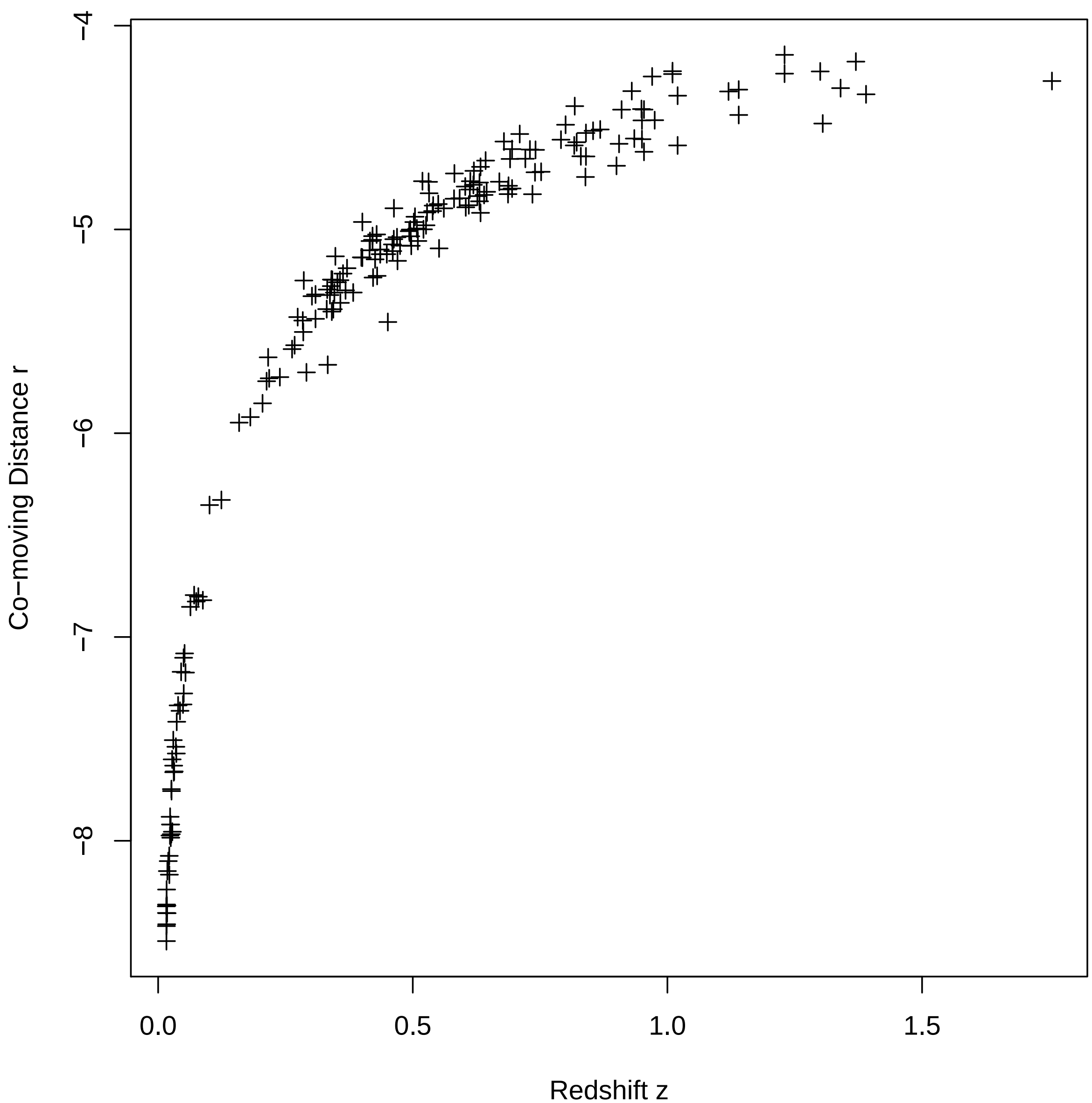
- The supernova data give us a way to infer the equation of state
- Observe distance modulus $(m - M)$ and redshift for a collection of type Ia SNe.
- Express in terms of co-moving distance r, assuming a flat universe.
- This yields

$$Y_i = \log_{10} r(z_i) + \sigma_i \epsilon_i, \qquad i = 1, \ldots, n,$$

  where r is the co-moving distance at each redshift z_i, and where the σ_i's are taken as known.

  We consider r "observable" because it can be directly estimated from the observed data.

The Data



Data from Davis et al 2007.

Connect to the Equation of State

The reconstruction equation

$$w(z) = \frac{H_0^2\Omega_m(1+z)^3 + \frac{2}{3}(1+z)r''(z)/(r'(z))^3}{H_0^2\Omega_m(1+z)^3 - 1/(r'(z))^2} - 1.$$

We can solve this differential equation to produce $r(z)$ as a functional of w with $r(0) = 0$ and

$$r(z) = H_0^{-1}\int_0^z ds\left[\Omega_{\rm M}(1+s)^3 + (1-\Omega_{\rm M})(1+s)^3\, e^{-3\int_0^s \frac{-w(u)}{1+u}\,du}\right]^{-\frac{1}{2}}$$

where H_0 is the Hubble constant and $\Omega_{\rm M}$ is the density of matter relative to the critical density.

This equation shows that inference for w from the supernova data Y is a nonlinear inverse problem.

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Inference for the Equation of State

- There is a variety of competing cosmological models that can be distinguished via inference from w. This is a key goal.
  - Cosmological constant ($w \equiv -1$)
  - Topological defect ($w \equiv -1/3$ or $w \equiv -2/3$)
  - Quintessence (various, roughly freezing $w'(z) > 0$, thawing $w'(z) < 0$)
  - Cardassian models ($w(0) < -1$)
  - ...
- Most methods for inferring w from Type Ia data fall into two groups:

  A. Infer $\rho_{\rm DE}/\rho_{\rm crit}$ using polynomial or nonparametric models for r and r'.

  B. Infer w from the reconstruction equation using polynomial or nonparametric models for r, r', r''.

  All of these require estimating at least one derivative. And derivative estimation is hard.

Derivative Estimation is Hard

We can think of derivative estimation as an ill-posed inverse problem.

Suppose we have data

$$Y_i = F(z_i) + \sigma_i \epsilon_i$$

and want to make inferences about $f \equiv F'$. Then we can write (in vector form)

$$Y = Kf + \Sigma^{1/2}\epsilon$$

where the operator $K = (K_1, \ldots, K_n)$ maps functions to $\mathbb{R}^n$ and where $K_i = \int_0^{z_i}$.

Create an orthonormal basis $\phi_1, \ldots, \phi_n$ from the eigenfunctions of K^*K with associated eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$.

Here, K^* is the adjoint of K given by

$$K^*u = \sum_{i=1}^{n} u_i 1_{[0,z_i]}.$$

Derivative Estimation (cont'd)

Then,

$$f = \sum_{j=1}^{n} \beta_j \phi_j + f_\perp$$

$$= \sum_{j=1}^{n} \lambda_j^{-1/2} \langle u_j, Kf \rangle \phi_j \, + \, f_\perp,$$

where $u_j = K\phi_j / \|K\phi_j\|$. The $f_\perp$ component is not estimable.

Using an optimal shrinkage scheme (cf. Donoho 1995),

$$MSE \approx \sum_{j=1}^{n} \mathrm{min}(\beta_j^2, \lambda_j^{-1}\tau_j^2),$$

where $\tau_j^2 = \sum_k u_{jk}^2 \sigma_k^2$.

Large components at high order are bad news!

A New Method

Instead of doing derivative estimation, we use the relationships between w and r

$$r = T(w; H_0, \Omega_{\mathrm{M}}) \equiv H_0^{-1} \int_0^z ds \left[\Omega_{\mathrm{M}}(1+s)^3 + (1-\Omega_{\mathrm{M}})(1+s)^3 e^{-3\int_0^s \frac{-w(u)}{1+u}\, du}\right]^{-\frac{1}{2}}$$

and treat it as a nonlinear inverse problem.

The structure of T gives us several useful features:

1. $r(0) = 0$
2. $(1+z)^{-3/2}/H_0 \le r'(z) \le (1+z)^{-3/2}/\sqrt{H_0^2 \Omega_m}$.

   In particular, r is monotone increasing.
3. When $w > -1/(1-\Omega_{\mathrm{M}})$, r is concave.
4. T is monotone (technically antitone) in w.

These allow us to use a variety of shape restrictions in our inferences.

Hypothesis Testing

Using shape restrictions, we can directly test a variety of hypotheses about w

A. Simple equalities for w: $w = w_0$,

B. Inequalities for w: $w_0 \leq w \leq w_1$,

C. Inequalities for w': $w'_0 \leq w' \leq w'_1$,

D. Inclusion: $w \in V$ for a linear space V, and

These hypotheses correspond directly to several cosmological models. For example, thawing models in quintessence must satisfy

$$\frac{1 + w(0)}{(1+z)^3} - 1 \leq w(z) \leq \frac{1 + w(0)}{1+z} - 1,$$

Hypothesis Testing: Method

The basic method

0. Select a small $0 < \alpha < 1$.
2. Construct a $1 - \alpha$ confidence set $\mathcal{C}$ for the unknown vector $(r(z_1), \ldots, r(z_n))$.
3. Construct the set R_0 of vectors $(r_0(z_1), \ldots, r_0(z_n))$ where r_0 is a co-moving distance function produced by an equation of state consistent with the null hypothesis
4. Reject the null hypothesis if $\mathcal{C} \cap R_0 = \emptyset$.

In practice, the sets in Steps 1 and 2 need not be constructed explicitly, and the procedure can be made computationally efficient for a broad range of hypotheses.

In Step 1, we adapt the shape-restricted confidence set from Davies et al (2007) and Baraud (2004).

Hypothesis Testing: Results

Current data do not allow sharp distinctions between models.

Model	Rejected At Level			
	32%	13%	5%	1%
Cosmological Constant	**yes**	no	no	no
Frustrated Cosmic Strings	**yes**	**yes**	**yes**	no
Domain Walls	**yes**	no	no	no
Nonaccelerating	**yes**	no	no	no
Quintessence Thawing	no	no	no	no
Quintessence Freezing	no	no	no	no
Constant w	no	no	no	no

The cosmological constant model cannot be ruled out with current data.

Note: These results are as good as what one gets under very optimistic assumptions (e.g., known parametric form).

Estimation

To estimate w, again use the relation $r = T(w; H_0, \Omega_{\mathrm{M}})$.

For a given parametric model for w, $w(z) = -\sum_j \beta_j \psi_j(z)$, we get a nonlinear, parametric form for r:

$$r(z) = H_0^{-1} \int_0^z ds \left[\Omega_m (1+s)^3 + (1-\Omega_m)(1+s)^3 \, e^{-3\sum_j \beta_j \widetilde{\psi}_j(s)}\right]^{-\frac{1}{2}},$$

where $\widetilde{\psi}_j(s) = \int_0^s \psi_j(u)/(1+u) du$.

This gives a likelihood over w, Ω_{M}, and H_0. We can fit this efficiently, and the results automatically satisfy the shape constraints.

Estimation (cont'd)

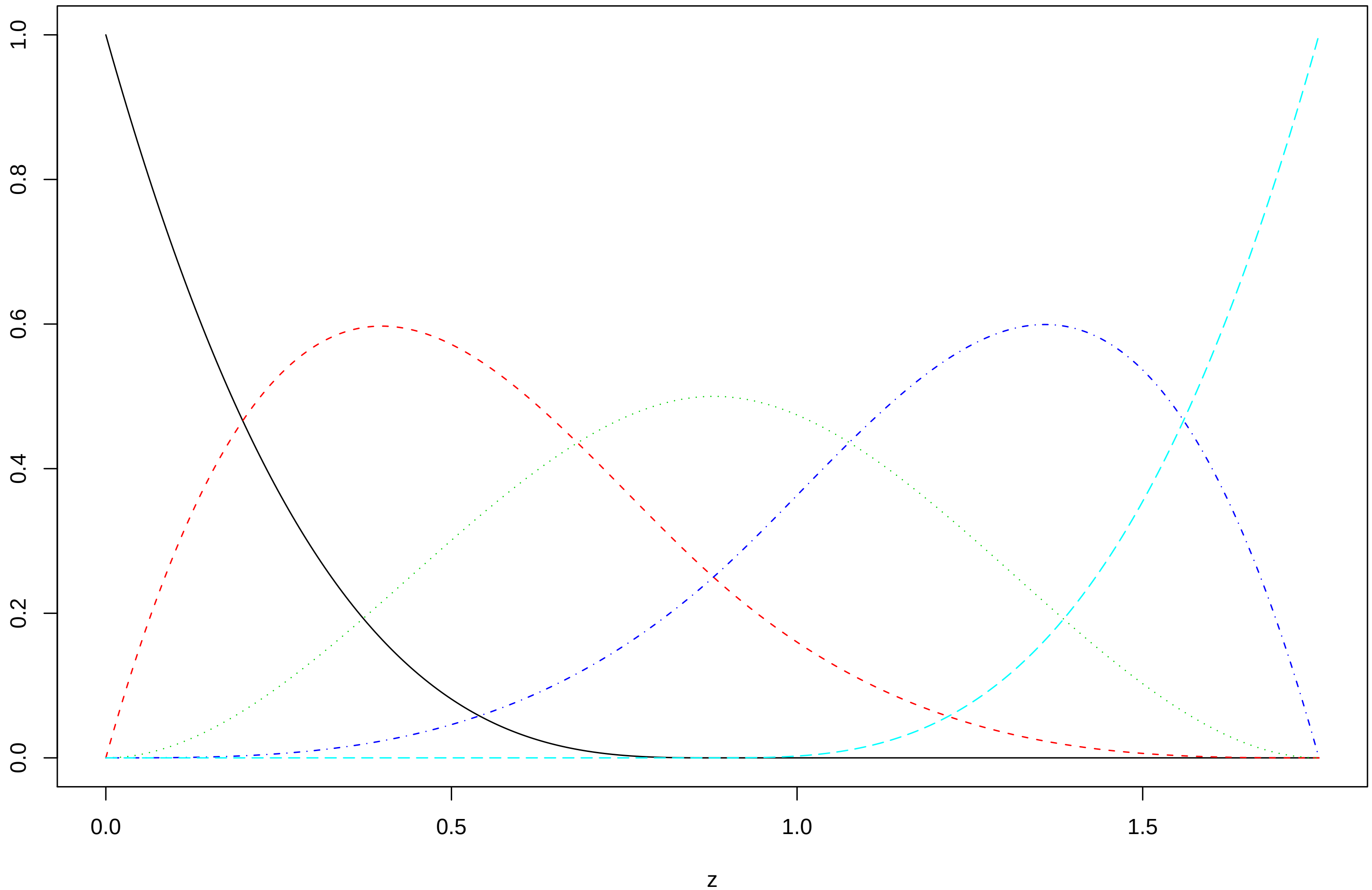
For a non-parametric analysis, we consider a collection $\mathcal{M}_1, \mathcal{M}_2, \ldots$ of parametric models of increasing dimension.

In all such models we can get reasonable fits, we then select an estimator by minimizing a measure of empirical risk (or alternatively by sequential testing).

Good confidence sets can be constructed under this scheme. Some theoretical details need to be better understood to get optimal performance.

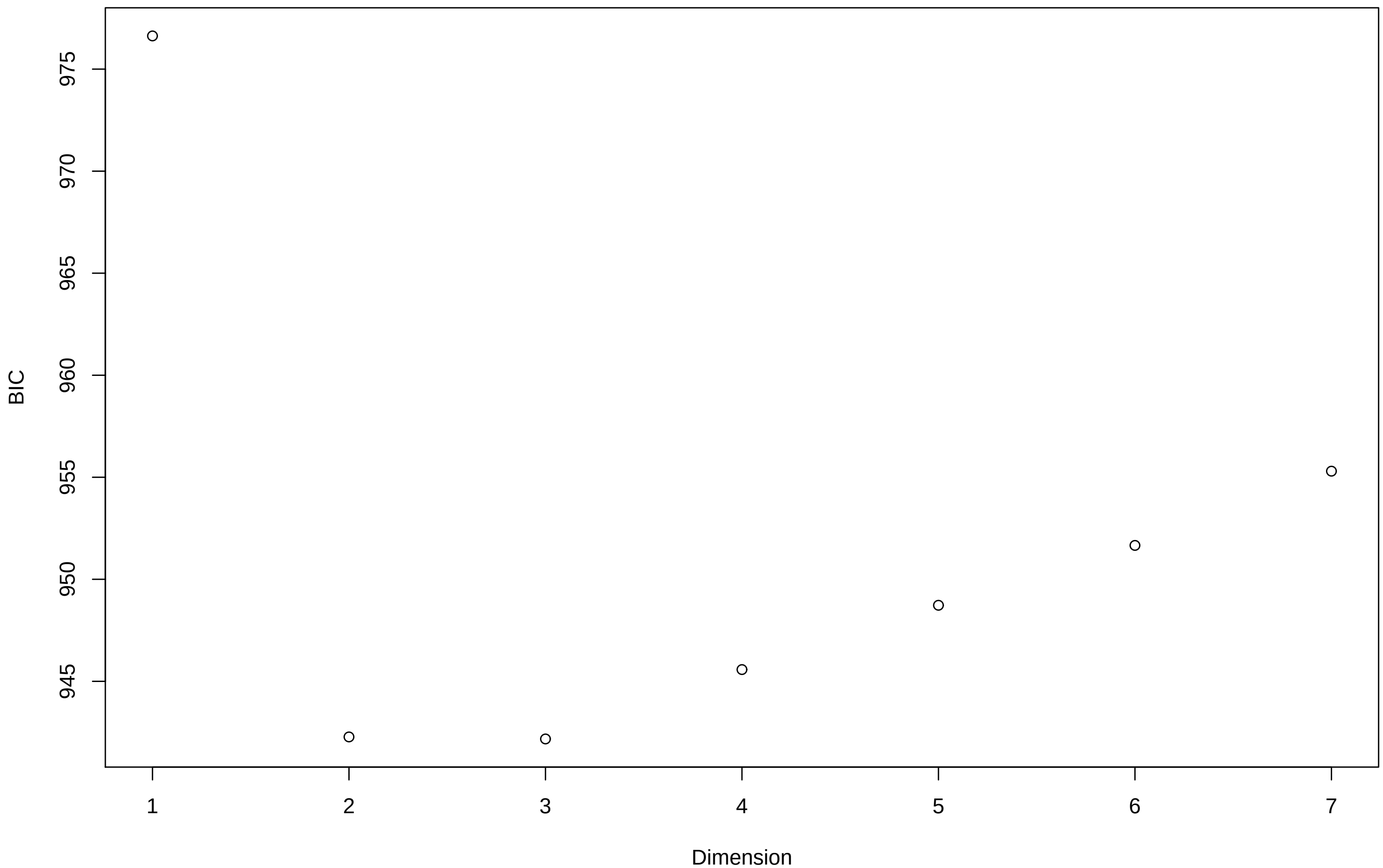
Example Basis

Even the low dimensional models cover the whole space



Dimensionality Estimated Well

Example simulation (this is typical):



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Fisher Follies

It is common practice in physics to use the eigenvectors of the model's Fisher Information matrix to construct a basis.

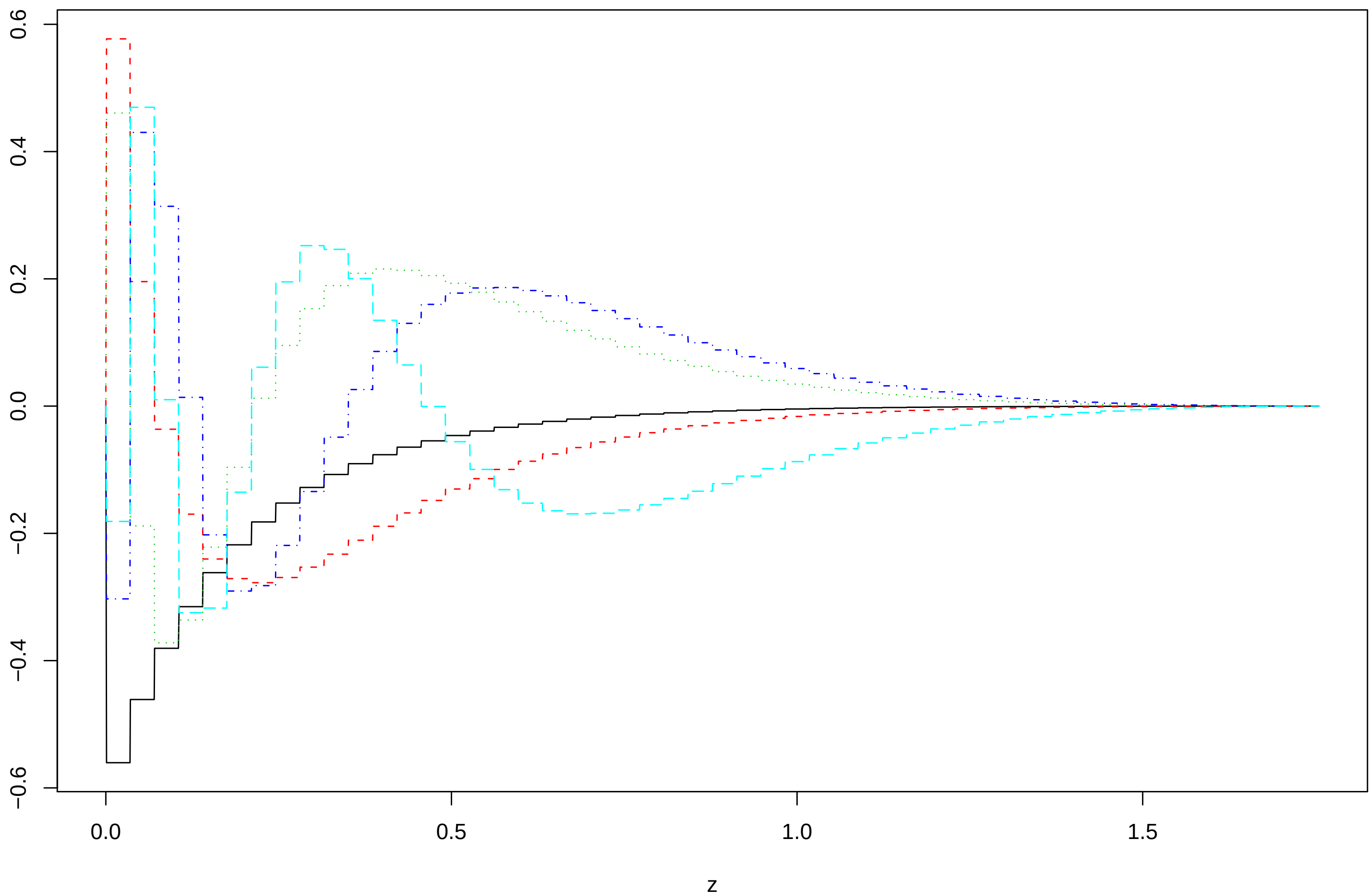
The k basis elements with the largest eigen values are selected where k can be chosen by some measure of empirical risk.

In an inverse problem like this, the Fisher basis can be very poor because it is not adapted to the forward mapping.

For T, structure at high z is poorly resolved. The Fisher basis requires large k to fit well, resulting in significant variance inflation.

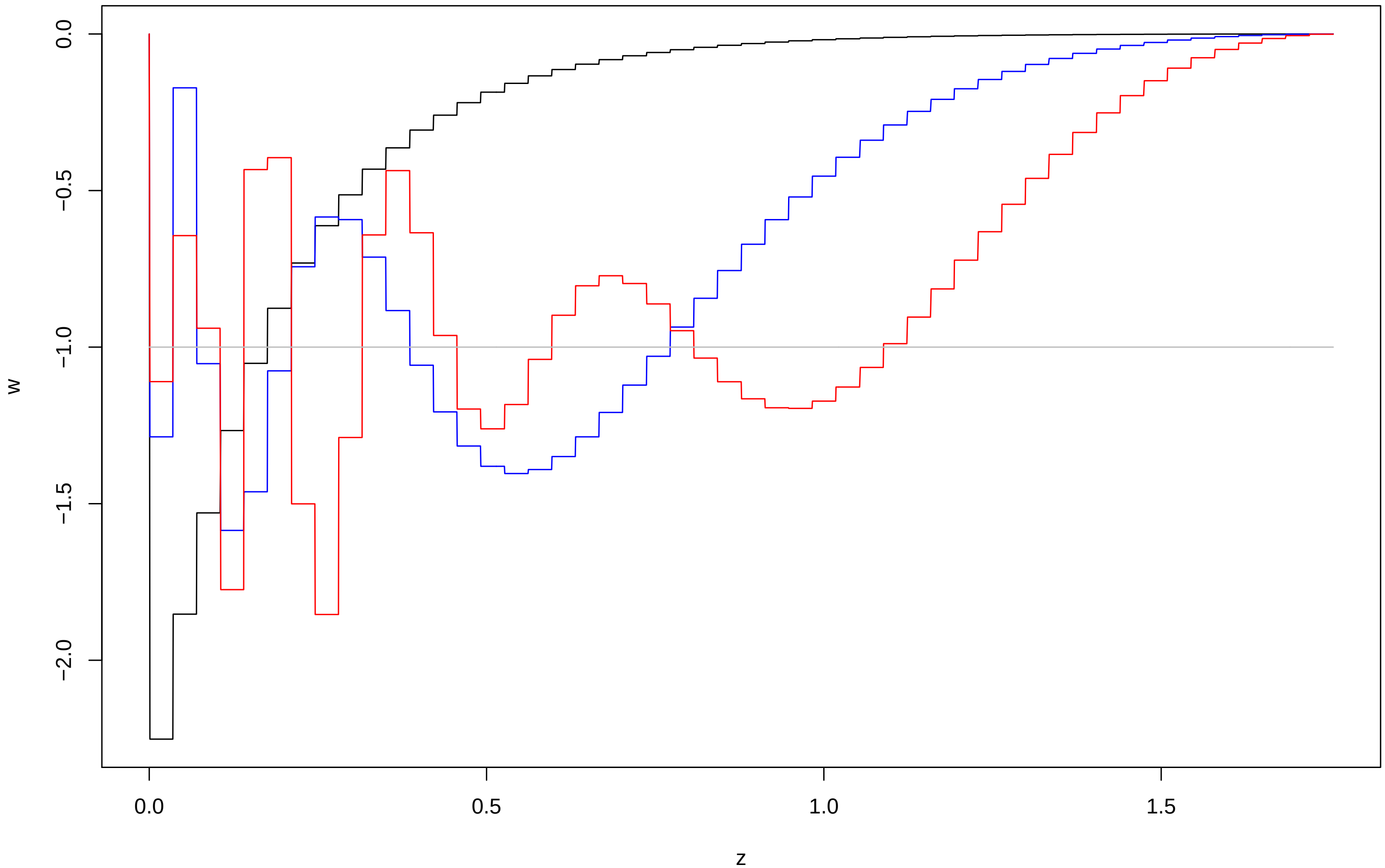
Fisher

Fisher basis elements with $k = 1, 5, 10$



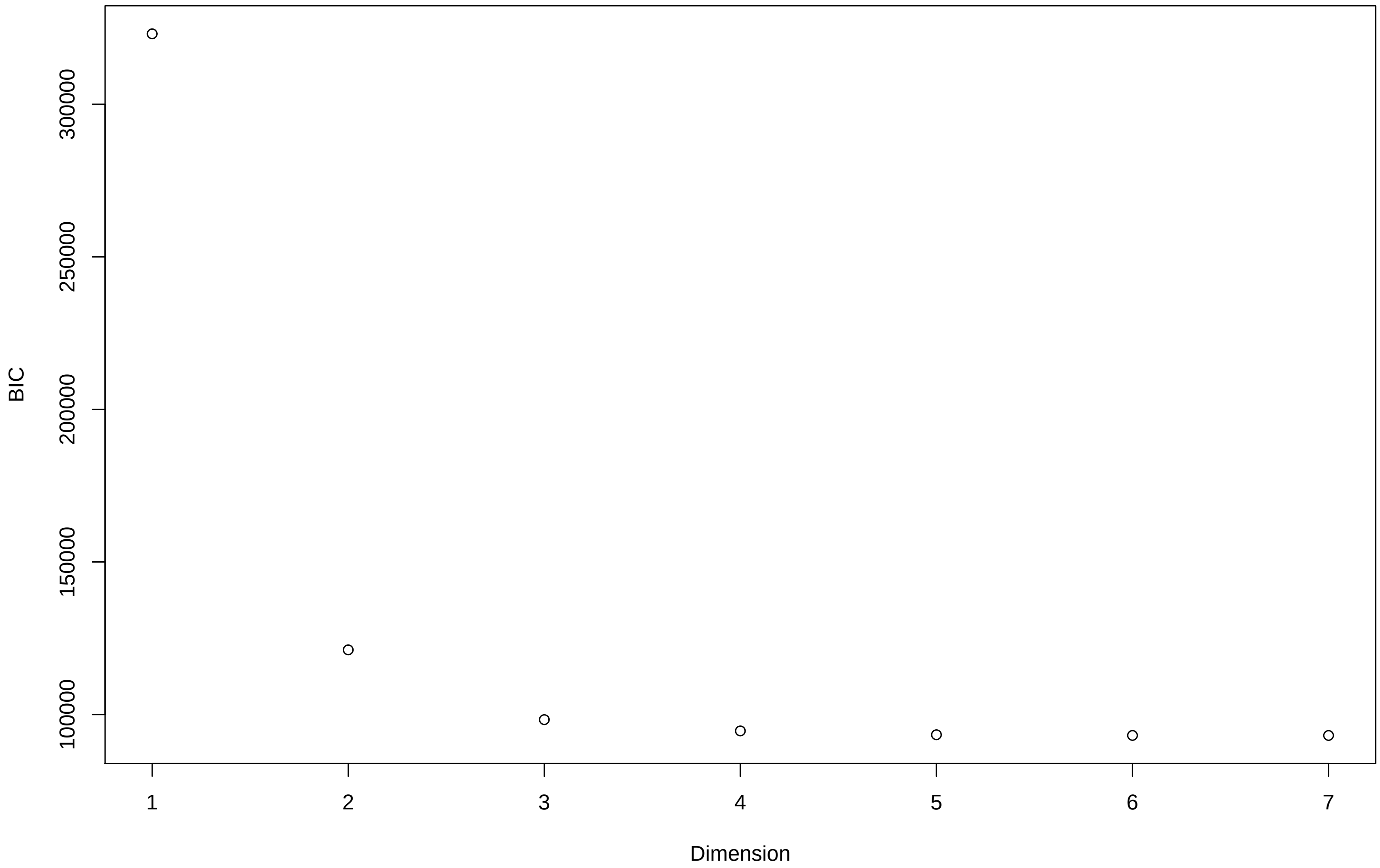
Fisher

Fisher best fits with $w \equiv -1$ and $k = 1, 5, 10$.



Fisher

Fisher dimensionality estimate, same simulation as above.



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Take-Home Points

- Nonparametric methods are essential here. We know very little a priori about w.

- Current data are insufficient to conclusively distinguish among competing models. This should change with new data coming down the pipeline.

  Nonetheless, cosmological constant model fits the current data well.

- Our can directly test a variety of interesting models with minimal assumptions and provide sharp nonparametric estimates for w.

- The Fisher basis need not be a good choice.