Cubic Splines

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1 References on splines

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2 Definition of spline

A spline is a piecewise polynomial in which the coefficients of each polynomial are fixed between 'knots' or joints.

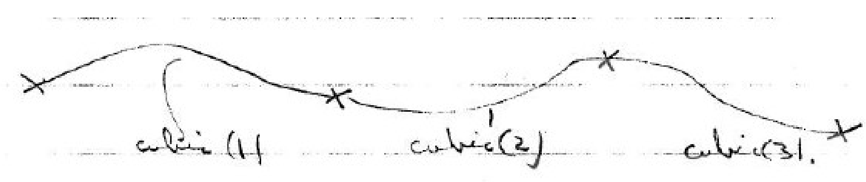

Figure 1:

Typically cubics are used. Then the coefficients are chosen to match the function and its first and second derivatives at each joint. There remain one free condition at each end, or two conditions at one end. However, using only starting conditions the spline is unstable. In general with n^{th} degree polynomials one can obtain continuity up to the $n-1$ derivative. The most common spline is a cubic spline. Then the spline function $y(x)$ satisfies $y^{(4)}(x) = 0$, $y(3)(x) = \text{const}$, $y''(x) = a(x) + h$. But for a beam between simple supports

$$y''(x) = \frac{M(x)}{EI}$$

where $M(x)$ varies linearly. Thus a spline is the curve obtained from a draughtsman's spline.

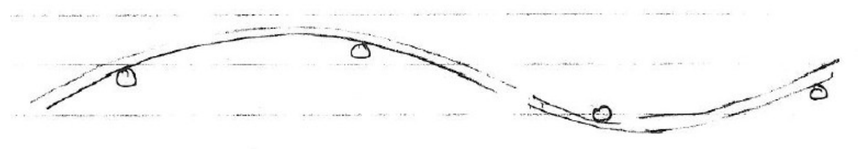

Figure 2: Draughtsman's spline

3 Equations of cubic spline

Let data be given at $x_0, x_1, \cdots x_n$ with values $y_0, y_1, \cdots y_n$. Let $S(x)$ bet the spline. Let

$$M_j = S''(x_j), \quad h_j = x_j - x_{j-1}$$

be 'moment' at j^{th} point. Then between x_{j-1} and x_j

$$S''(x) = M_{j-1}\frac{x_j - x}{h_j} + M_j\frac{x - x_{j-1}}{h_j}$$

$$S'(x) = -M_{j-1}\frac{(x_j - x)^2}{2h_j} + M_j\frac{(x - x_{j-1})^2}{2h_j} + A$$

$$S(x) = M_{j-1}\frac{(x_j - x)^3}{6h_j} + M_j\frac{(x - x_{j-1})^3}{6h_j} + Ax + B$$

But

$$S(x_{j-1}) = y_{j-1}, \quad S(x_j) = y_j.$$

We obtain

$$S(x) = M_{j-1}\frac{(x_j - x)^3}{6h_j} + M_j\frac{(x - x_{j-1})^3}{6h_j} + \left(y_{j-1} - M_{j-1}\frac{h_j^2}{6}\right)\frac{x_j - x}{h_j} + \left(y_j + M_j\frac{h_j^2}{6}\right)\frac{x - x_j}{h_j}$$

$$S'(x) = -M_{j-1}\frac{(x_j - x)^2}{2h_j} + M_j\frac{(x - x_{j-1})^2}{2h_j} + \frac{y_j - y_{j-1}}{h_j} - \frac{M_j - M_{j-1}}{6}h_j$$

We now have continuity of $S''(x)$ and $S(x)$ at x_j. We also require continuity of $S'(x)$. Now

$$S'(x_j^-) = \frac{h_j}{6}M_{j-1} + \frac{h_j}{3}M_j + \frac{y_j - y_{j-1}}{h_j}$$

$$S'(x_j^+) = \frac{-h_{j+1}}{3}M_j - \frac{h_{j+1}}{6}M_{j+1} + \frac{y_{j+1} - y_j}{h_{j+1}}$$

Equating these gives equations for the M_j at every interior node:

$$\frac{h_j}{6}M_{j-1} + \frac{h_j + h_{j+1}}{3}M_j + \frac{h_{j+1}}{6}M_{j+1} = \frac{y_{j+1} - y_j}{h_{j+1}} - \frac{y_j - y_{j-1}}{h_j}$$

To complete the system we need 2 additional equations. Setting $S''(x_0) = M_0 = 0$, $S''(x_n) = M_n = 0$, corresponds to simple supports at the ends of the beam. Better accuracy is obtained if $S'(x_0)$ and $S'(x_n)$ are known. Then from the equation for $S'(x_n)$ we have

$$\frac{h_1}{3}M_0 + \frac{h_1}{6}M_1 = \frac{y_1 - y_0}{h_1} - y_0'$$

$$\frac{h_n}{6}M_{n-1} + \frac{h_n}{3}M_n = y_n' - \frac{y_n - y_{n-1}}{h_n}$$

If neither $S'(x)$ nor $S''(x)$ are known at the ends, one may set $S'''(x) = 0$, or

$$M_1 - M_0 = 0$$

$$M_{n-1} - M_n = 0$$

Alternatively one may use some linear combination of these end conditions. In any case one obtains a tridiagonal set of equations which may be written

$$\begin{bmatrix} 2 & \lambda_0 & & & \\ \mu_1 & 2 & \lambda_1 & & \\ & & & & \\ & & \mu_{n-1} & 2 & \lambda_{n-1} \\ & & & \mu_n & 2 \end{bmatrix} \begin{bmatrix} M_0 \\ M_1 \\ \\ M_{n-1} \\ M_n \end{bmatrix} = \begin{bmatrix} d_0 \\ d_1 \\ \\ d_{n-1} \\ d_n \end{bmatrix}$$

where at interior points

$$\lambda_j = \frac{h_{j+1}}{h_j + h_{j+1}}, \qquad \mu_j = 1 - \lambda_j$$

$$d_j = 6\frac{(y_{j+1} - y_j)/h_{j+1} - (y_j - y_{j-1})/h_j}{h_j + h_{j+1}}$$

and λ_0, μ_n, d_0, d_n depend on the end conditions. These equations are diagonally dominant, leading to stable behaviour.

Spline equations for first derivatives

Let

$$m_j = S'(x_j)$$

Then

$$\begin{aligned} S(x) = \quad & m_{j-1}\frac{(x_j - x)^2(x - x_{j-1})}{h_j^2} - m_j\frac{(x - x_{j-1})^2(x_j - x)}{h_j^2} \\ & + y_{j-1}\frac{(x_j - x)^2\left[2(x - x_{j-1}) + h_j\right]}{h_j^3} + y_j\frac{(x - x_{j-1})^2\left[2(x_j - x) + h_j\right]}{h_j^3} \end{aligned}$$

$$S'(x) = \quad m_{j-1}\frac{(x_j - x)(2x_{j-1} + x_j - 3x)}{h_j^2} - m_j\frac{(x - x_{j-1})(2x_j - x_{j-1} - 3x)}{h_j^2}$$
$$+6\frac{y_j - y_{j-1}}{h_j^3}(x_j - x)(x - x_{j-1})$$

$$S''(x) = \quad -2m_{j-1}\frac{2x_j + x_{j-1} - 3x}{h_j^2} - 2m_j\frac{2x_{j-1} + x_j - 3x)}{h_j^2}$$
$$+6\frac{y_j - y_{j-1}}{h_j^3}(x_j + x_{j-1} - 2x)$$

$$S''(x_j^-) = \frac{2m_{j-1}}{h_j} + \frac{4m_j}{h_j} - 6\frac{y_j - y_{j-1}}{h_j^2}$$

$$S''(x_j^+) = -\frac{4m_j}{h_{j+1}} - \frac{2m_{j+1}}{h_{j+1}} + 6\frac{y_{j+1} - y_j}{h_{j+1}^2}$$

Equating these gives

$$\frac{1}{h_j}m_{j-1} + 2\left(\frac{1}{h_j} + \frac{1}{h_{j+1}}\right)m_j + \frac{1}{h_{j+1}}m_{j+1} = 3\frac{y_j - y_{j-1}}{h_j^2} + 3\frac{y_{j+1} - y_j}{h_{j+1}^2}$$

Then

$$\begin{bmatrix} 2 & \mu_0 & & & & \\ \lambda_1 & 2 & \mu_1 & & & \\ & & & & & \\ & & & \lambda_{N-1} & 2 & \mu_{N-1} \\ & & & & \lambda_N & 2 \end{bmatrix} \begin{bmatrix} m_0 \\ m_1 \\ \\ m_{N-1} \\ m_N \end{bmatrix} = \begin{bmatrix} c_0 \\ c_1 \\ \\ c_{N-1} \\ c_N \end{bmatrix}$$

where at interior points

$$c_j = 3\lambda_j\frac{y_j - y_{j-1}}{h_j} + 3\mu_j\frac{y_{j+1} - y_j}{h_{j+1}}$$

The end condition $S'' = 0$ gives

$$\mu_0 = 1, \quad c_0 = 3\frac{y_j - y_0}{h_j}$$

$$\lambda_N = 1, \quad c_N = 3\frac{y_N - y_{N-1}}{h_{N-1}}$$

4 Minimum curvature property of splines

Consider all functions $f(x)$ such that $f(x_i) = y_i$ Then the spline $S(x)$ with $S''(x_0) = S''(x_n) = 0$ is that function $f(x)$ which minimize:

$$\int_{x_0}^{x_n} f''(x)^2 dx$$

(Ahlberg, p. 76)

Proof

$$\int_{x_0}^{x_n} \left(f''(x) - S''(x)\right)^2 dx$$

$$= \int_{x_0}^{x_n} f''(x)^2 dx - 2\int \left(f''(x) - S''(x)\right) S''(x) dx - \int S''(x)^2 dx$$

The middle term is

$$\sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} \left(f''(x) - S''(x)\right) S''(x) dx$$

Integrating by parts it becomes

$$\sum_{i=1}^{n} \left[\left(f'(x) - S'(x)\right) S''(x)\right]_{i-1}^{i} - \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} \left(f'(x) - S'(x)\right) S'''(x) dx$$

$S'''(x)$ is constant in each segment, and can be taken out of each integral in the second term.

But

$$\int_{x_{i-1}}^{x_i} (f'(x) - S'(x))\, dx = [f(x) - S(x)]_{i-1}^{i}$$

which vanishes because

$$f(x_i) = S(x_i) = y_i, \quad i = 0, 1, \ldots n$$

Also the contributions to the sum in the first term vanish at every interior node, leaving

$$(f'(x_n) - S'(x_n))\, S''(x_n) - (f'(x_0) - S'(x_0))\, S''(x_0)$$

Thus

$$\int_{x_0}^{x_n} (f''(x) - S''(x))\, S''(x) dx = 0$$

if either $S''(x) = 0$ at each end or $f'(x) = S'(x)$. Consequently

$$\int_{x_0}^{x_n} f''(x)^2 dx - \int_{x_0}^{x_n} S''(x)^2 dx = \int_{x_0}^{x_n} (f''(x) - S''(x))^2\, dx > 0$$

unless $f(x) = S(x)$.

5 Errors in spline approximation

If the maximum interval is h then

$$f(x) - S(x) \sim h^4$$

$$f'(x) - S'(x) \sim h^3$$

$$f''(x) - S''(x) \sim h^2$$

$$f'''(x) - S'''(x) \sim h$$

This is proved by Ahlberg, Nilson and Walsh, p. 29, provided that $f^{(4)}$ is continuous.

To study convergence properties of splines we need an estimate of the norm of the inverse of the matrix B of the equations for the spline. Taking

$$\|x\| = \max|x_i|$$

the induced norm of B is

$$\|B\| = \sup_{x \neq 0} \frac{\|Bx\|}{\|x\|} = \max_i \sum_{j=1}^{n} |b_{ij}|$$

With a diagonally dominant matrix we have

$$\begin{aligned} \|y\| = \|Bx\| &= \max_i \left| \sum_{j=1}^{n} b_{ij} x_j \right| \\ &\geq \left| \sum_{j=1}^{n} b_{ij} x_j \right| \end{aligned}$$

where $\|x\| = |x_k|$, or

$$\begin{aligned} \|y\| &\geq \left\{ |b_{kk}| - \sum_{j \neq k} |b_{kj}| \right\} \|x\| \\ &\geq \min_i \left\{ |b_{ii}| - \sum_{j \neq i} |b_{ij}| \right\} \|x\| \end{aligned}$$

Thus

$$\|B^{-1}\| = \sup_{y \neq 0} \frac{\|B^{-1}y\|}{\|y\|} \leq \left[\min_i \left\{ |b_i i| - \sum_{j \neq i} |b_{ij}| \right\} \right]^{-1}$$

or for the spline matrix

$$\|B^{-1}\| \leq \max \left[(2 - \lambda_0)^{-1}, (2 - \lambda_n)^{-1}, 1 \right]$$

Now let

$$\delta = \max h_j$$

and let

$$\max \frac{\delta}{h_j} \leq \beta < \infty.$$

Then the spline has the following convergence propety as the maximum mesh spacing

δ is reduced:

<u>Theorem A:</u> (Ahlberg, p. 22)

Let $f(x)$ be continuous on $[a, b]$ and let $\max\left[|\lambda_0|, |\mu_n|\right] < 2$. Then if $S(x)$ is a spline fit to $f(x)$ with $x_0 = a$, $x_n = b$, then

$$|f(x) - S(x)| \to 0 \quad \text{as} \quad \delta \to 0.$$

Also if

$$|f(x) - f(x')| \leq \kappa \left|x - x'\right|^{\alpha}, \quad 0 \leq \alpha \leq 1$$

then

$$|f(x) - S(x)| = O\left(\delta^{\alpha}\right).$$

<u>Proof:</u> On $[x_{j-1}, x_j]$ the spline equations can be arranged as

$$S(x) - f(x) = \frac{M_{j-1}(x_j - x)\left[(x_j - x)^2 - h_j^2\right]}{6h_j} + \frac{M_j(x - x_{j-1})\left[(x - x_{j-1})^2 - h_j^2\right]}{6h_j}$$

$$+\frac{f_j + f_{j-1}}{2} - f(x) + (f_j - f_{j-1})\frac{x_j + x_{j-1} - 2x}{2h_j}$$

By maximizing the coefficients of M_j and M_{j-1} we find that they do not exceed $h_j^2/3^{5/2}$. Let A_ij be the elements of B^{-1} where B is the coefficient matrix. Then

$$M_j = 6\sum_{i=1}^{n-1} A_{ji}\frac{(f_{i+1} - f_i)/h_{i+1} - (f_i - f_{i-1})/h_i}{h_i + h_{i+1}} + A_{j0}d_0 + A_{jn}d_n.$$

Also let $\mu(f,\delta)$ be the modulus of continuity of $f(x)$ on $[a,b]$ defined as

$$\mu(f,\delta) = \sup_{|x-x'|<\delta} |f(x) - f(x')|$$

Then remembering the condition that $\delta \leq \beta h_j$, we find that

$$h_j^2 \left| \frac{(f_{i+1} - f_i)/h_{i+1} - (f_i - f_{i-1})/h_i}{h_i + h_{i+1}} \right| \leq \beta^2 \mu(f,\delta)$$

Thus

$$h_j^2 \left(|M_{j-1} + |M_j| \right) \leq 2\|B^{-1}\| \left\{ 6\beta^2 \mu(f,\delta) + \delta^2(|d_0| + |d_n|) \right\}$$

and finally

$$|S(x) - f(x)| \leq \frac{2}{3^{\frac{5}{2}}} \|B^{-1}\| \left\{ 6\beta^2 \mu(f,\delta) + \delta^2(|d_0| + |d_n|) \right\} + \mu(f,\delta) \left(1 + \frac{1}{2} \right)$$

If the function satisfies

$$|f(x) - f(x')| \leq \kappa (x - x')^\alpha, \quad 0 < \alpha \leq 1$$

then

$$\mu(f,\delta) \leq \kappa \delta^\alpha$$

establishing the required result, in light of the inequality satisfied by $\|B^{-1}\|$, provided that $\lambda < 2, \mu_n < 2$.

If f is in C^2 we can prove

<u>Theorem B:</u> (Ahlberg, p. 27)

$$|f^{(p)}x - S^{(p)}(x)| = O(\delta^{2-p})$$

where $S(x)$ satisfies end conditions

$$2M_0 + M_1 = \frac{6}{h_1}\left(\frac{f_1 - f_0}{h_1} - f_0'\right)$$

or

$$M_0 = f_0''.$$

Proof: If B is the matrix of the spline we have

$$BM = d$$

whence

$$B\left(M - \frac{d}{3}\right) = \left(I - \frac{B}{3}\right)d$$

The right side is

$$\frac{1}{3}\begin{bmatrix} d_0 - d_1 \\ \mu_1(d_1 - d_0) - \lambda_1(d_2 - d_1) \\ \cdots \\ d_N - d_{N-1} \end{bmatrix}$$

Now at interior points

$$\begin{aligned} \frac{d_j}{6} &= \frac{(f_{j+1} - f_j)/h_{j+1} - (f_j - f_{j-1})/h_j}{h_j + h_{j+1}} = f[x_{j-1}, x_j, x_{j+1}] \\ &= \frac{1}{2}f''(\xi_j), \quad x_{j-1} \le \xi_j \le x_{j+1} \end{aligned}$$

Also by the Taylor theorem

$$\frac{d_0}{6} = \frac{(f_1 - f_0)/h_1 - f_0'}{h_1} = \frac{1}{2} f''(\xi_0), \quad x_0 \leq \xi_0 \leq x_1$$

Thus if μ is the modulus of continuity of f'',

$$|f''(x) - f''(x')| \leq \mu(f'', \delta), \quad |x - x'| \leq \delta$$

then here since $\lambda_j + \mu_j = 1$,

$$\left\| \left(I - \frac{B}{3} \right) d \right\| \leq 3\mu(f'', \delta)$$

Also with the given end conditions $\|B^{-1}\| \leq 1$ so

$$\left\| M - \frac{d}{3} \right\| \leq \|B^{-1}\| \left\| \left(I - \frac{B}{3} \right) d \right\| \leq 3\mu(f'', \delta)$$

Now since $\dfrac{d_j}{3} = f''(\xi_j)$ it also follows that

$$\left\| f'' - \frac{d}{3} \right\| \leq \mu(f'', \delta)$$

whence

$$\|M - f''\| \leq 4\mu(f'', \delta)$$

and since S'' is linear on each interval

$$\|S'' - f''\| \leq 5\mu(f'', \delta)$$

Now if

$$|f''(x) - f''(x')| \le \kappa |x - x'|^\alpha, \quad 0 < \alpha \le 1$$

we have

$$\|S'' - f''\| \le 5\kappa\delta^\alpha$$

Since $S(x_j) = f(x_j)$, by Rolle's theorem, there is a point ξ_j in the interval $[x_{j-1}, x_j]$ at which $S'(\xi_j) = f'(\xi_j)$. Then on this interval

$$|f'(x) - S'(x)| = \left| \int_{\xi_j}^{x} \left(f''(x) - S''(x)\right) \right| \le 5\delta\mu(f'', \delta)$$

and integrating a second time

$$|f(x) - S(x)| \le \frac{5}{2}\delta^2\mu(f'', \delta)$$

Even stronger convergence theorems can be proved. Ahlberg (p. 29) gives:

<u>Theorem C:</u> Let $f^{(4)}(x)$ be continuous then

$$f^{(p)}(x) - S^{(p)}(x) = O\left(\delta^{3-p}\right), \quad p = 0, 1, 2, 3$$

<u>B spline</u>

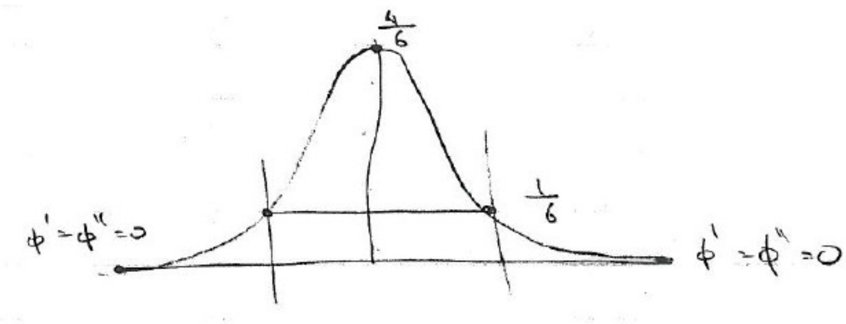

Figure 3: B spline

6 Smoothing splines

Instead of requiring a spline to coincide with the function at specidied points one may simply require it to pass closer and permit some smoothing. To do this let $S(x)$ be chosen to minimize

$$R_1 \int_{x_0}^{x_n} f''(x)^2 dx + R_2 \sum_{i=0}^{n} \frac{(f(x_i) - y_i)^2}{Q_i}$$

Then by the calculus of variations the solution can be shown to be a spline with joints at $x_1, x_2, \cdots x_{n-1}$ which does not necessarily pass through thte points.

Reinsch (Numer. Math 10, 1967, pp. 177-183) shows that the equations now reduce to a five diagonal system, which can still be solved with a number of operations proportional to n.

7 Change of independent variable

With any type of polynomial fitting or spline fitting difficulties occur if the function $y(x)$ has too large a slope. In extreme cases the given points may represent a closed curve or other curve that bends back on itself.

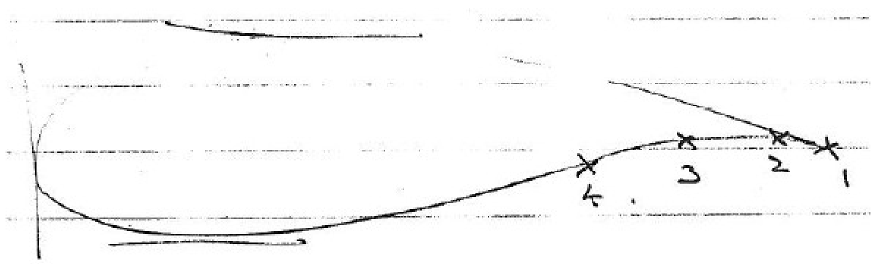

Figure 4:

In such a case the x_i do not form a monotone sequence, and also $\frac{dy}{dx}$ is inifinite at the nose, while $\frac{dx}{dy}$ is infinite at the top and bottom. To overcome this difficulty we need to parameterize the curve and set

$$x = x(t)$$

$$y = y(t)$$

where t is a new independent variable that increases monotonely. Then form separate splines for $x(t)$ and $y(t)$.

8 Integration

Splines are useful for integration when function is defined by a table of coordinates at unequal intervals.

Conclusions

Acknowledgements

References