Web Solutions for

How to Read and Do Proofs

An Introduction to Mathematical Thought Processes

Fifth Edition

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1.5 a. Hypothesis: A, B and C are sets of real numbers with $A \subseteq B$.
Conclusion: $A \cap C \subseteq B \cap C$.

b. Hypothesis: For a positive integer n, the function f defined by:

$$f(n) = \begin{cases} n/2, & \text{if } n \text{ is even} \\ 3n+1, & \text{if } n \text{ is odd} \end{cases}$$

For an integer $k \geq 1$, $f^k(n) = f^{k-1}(f(n))$, and $f^1(n) = f(n)$.
Conclusion: For any positive integer n, there is an integer $k > 0$ such that $f^k(n) = 1$.

c. Hypothesis: x is a real number.
Conclusion: The minimum value of $x(x-1) \geq -1/4$.

1.14 (T = true, F = false)

A	B	C	$A \Rightarrow B$	$(A \Rightarrow B) \Rightarrow C$
T	T	T	T	T
T	T	F	T	F
T	F	T	F	T
T	F	F	F	T
F	T	T	T	T
F	T	F	T	F
F	F	T	T	T
F	F	F	T	F

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Web Solutions to Exercises

2.1 The forward process makes use of the information contained in the hypothesis A. The backward process tries to find a chain of statements leading to the fact that the conclusion B is true.

With the backward process, you start with the statement B that you are trying to conclude is true. By asking and answering key questions, you derive a sequence of new statements with the property that if the sequence of new statements is true, then B is true. The backward process continues until you obtain the statement A or until you can no longer ask and/or answer the key question.

With the forward process, you begin with the statement A that you assume is true. You then derive from A a sequence of new statements that are true as a result of A being true. Every new statement derived from A is directed toward linking up with the last statement obtained in the backward process. The last statement of the backward process acts as the guiding light in the forward process, just as the last statement in the forward process helps you choose the right key question and answer.

2.4 (c) is incorrect because it uses the specific notation given in the problem.

2.18 a. Show that the two lines do not intersect.
Show that the two lines are both perpendicular to a third line.
Show that the two lines are both vertical or have equal slopes.
Show that the two lines are each parallel to a third line.
Show that the equations of the two lines are identical or have no common solution.

b. Show that their corresponding side-angle-sides are equal.
Show that their corresponding angle-side-angles are equal.
Show that their corresponding side-side-sides are equal.
Show that they are both congruent to a third triangle.

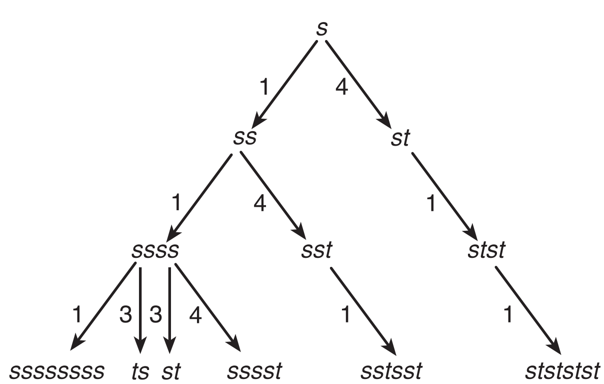
2.23 (1) How can I show that a triangle is equilateral?
(2) Show that the three sides have equal length (or show that the three angles are equal).
(3) Show that $\overline{RT} = \overline{ST} = \overline{SR}$ (or show that $\angle R = \angle S = \angle T$).

2.34 For sentence 1: The fact that $c^n = c^2c^{n-2}$ follows by algebra. The author then substitutes $c^2 = a^2 + b^2$, which is true from the Pythagorean theorem applied to the right triangle.

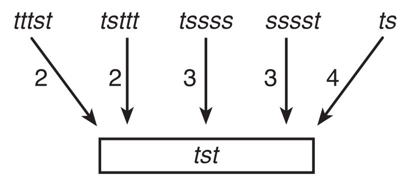
For sentence 2: For a right triangle, the hypotenuse c is longer than either of the two legs a and b so, $c > a$, $c > b$. Because $n > 2$, $c^{n-2} > a^{n-2}$ and $c^{n-2} > b^{n-2}$ and so, from sentence 1, $c^n = a^2c^{n-2} + b^2c^{n-2} > a^2(a^{n-2}) + b^2(b^{n-2})$.

For sentence 3: Algebra from sentence 2.

2.39 a. The number to the left of each line in the following figure indicates which rule is used.



b. The number to the left of each line in the following figure indicates which rule is used.



c.

$A:$	s	given
$A1:$	ss	rule 1
$A2:$	$ssss$	rule 1
$B1:$	$sssst$	rule 4
$B:$	tst	rule 3

2.42 **Analysis of Proof.** A key question associated with the conclusion is, "How can I show that a triangle is equilateral?" One answer is to show that all three sides have equal length, specifically,

$B1: \ \overline{RS} = \overline{ST} = \overline{RT}.$

To see that $\overline{RS} = \overline{ST}$, work forward from the hypothesis to establish that

$B2:$ Triangle RSU is congruent to triangle SUT.

Specifically, from the hypothesis, SU is a perpendicular bisector of RT, so

$A1: \ \overline{RU} = \overline{UT}.$

In addition,

$A2: \ \angle RUS = \angle SUT = 90^o.$
$A3: \ \overline{SU} = \overline{SU}.$

Thus the side-angle-side theorem states that the two triangles are congruent and so $B2$ has been established.

It remains (from $B1$) to show that

$B3: \ \overline{RS} = \overline{RT}.$

Working forward from the hypothesis you can obtain this because

$A4: \ \overline{RS} = 2\overline{RU} = \overline{RU} + \overline{UT} = \overline{RT}.$

Proof. To see that triangle RST is equilateral, it will be shown that $\overline{RS} = \overline{ST} = \overline{RT}$. To that end, the hypothesis that SU is a perpendicular bisector of RT ensures (by the side-angle-side theorem) that triangle RSU is congruent to triangle SUT. Hence, $\overline{RS} = \overline{ST}$. To see that $\overline{RS} = \overline{RT}$, by the hypothesis, one can conclude that $\overline{RS} = 2\overline{RU} = \overline{RU} + \overline{UT} = \overline{RT}$. □

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Web Solutions to Exercises

3.4 (A is the hypothesis and $A1$ is obtained by working forward one step.)

a. A : n is an odd integer.
$A1$: $n = 2k + 1$, where k is an integer.

b. A : s and t are rational numbers with $t \neq 0$.
$A1$: $s = p/q$, where p and q are integers with $q \neq 0$. Also, $t = a/b$, where $a \neq 0$ and $b \neq 0$ are integers.

c. A : $\sin(X) = \cos(X)$.
$A1$: $x/z = y/z$ (or $x = y$).

d. A : a, b, c are integers for which $a|b$ and $b|c$.
$A1$: $b = pa$ and $c = qb$, where p and q are both integers.

3.7 (T = true, F = false)

a. Truth Table for "*A AND B*."

A	B	"*A AND B*"
T	T	T
T	F	F
F	T	F
F	F	F

b. Truth Table for "A *AND NOT* B"

A	B	*NOT* B	"A *AND NOT* B"
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

3.14 a. If the four statements in part (a) are true, then you can show that A is equivalent to any of the alternatives by using Exercise 3.13. For instance, to show that A is equivalent to D, you already know that "D implies A." By Exercise 3.13, because "A implies B," "B implies C," and "C implies D," you have that "A implies D."

b. The advantage of the approach in part (a) is that only four proofs are required ($A \Rightarrow B$, $B \Rightarrow C$, $C \Rightarrow D$, and $D \Rightarrow A$) as opposed to the six proofs ($A \Rightarrow B$, $B \Rightarrow A$, $A \Rightarrow C$, $C \Rightarrow A$, $A \Rightarrow D$, and $D \Rightarrow A$) required to show that A is equivalent to each of the three alternatives.

3.20 **Analysis of Proof.** The key question for this problem is, "How can I show that a triangle is isosceles?" This proof answers this question by recognizing that the conclusion of Proposition 3 is the same as the conclusion you are trying to reach. So, if the current hypothesis implies that the hypothesis of Proposition 3 is true, then the triangle is isosceles. Because triangle UVW is a right triangle, on matching up the notation, all that remains to be shown is that $\sin(U) = \sqrt{u/2v}$ implies $w = \sqrt{2uv}$. To that end,

$A:\ \sin(U) = \sqrt{\frac{u}{2v}}$ (by hypothesis)

$A1:\ \sin(U) = \frac{u}{w}$ (by definition of sine)

$A2:\ \frac{u}{w} = \sqrt{\frac{u}{2v}}$ (from A and $A1$)

$A3:\ \frac{u}{2v} = \frac{u^2}{w^2}$ (square both sides of $A2$)

$A4:\ uw^2 = 2vu^2$ (cross-multiply $A3$)

$A5:\ w^2 = 2vu$ (divided $A4$ by u)

$B:\ w = \sqrt{2uv}$ (take the square root of both sides of $A5$)

It has been shown that the hypothesis of Proposition 3 is true, so the conclusion of Proposition 3 is also true. Hence triangle UVW is isosceles.

3.26 **Analysis of Proof.** The forward-backward method gives rise to the key question, "How can I show that a triangle is isosceles?" Using the definition of an isosceles triangle, you must show that two of its sides are equal, which, in this case, means you must show that

$$B1 : \; u = v.$$

Working forward from the hypothesis, you have the following statements and reasons

Statement	**Reason**
$A1 : \sin(U) = \sqrt{u/2v}$.	Hypothesis.
$A2 : \sqrt{u/2v} = u/w$.	Definition of sine.
$A3 : w^2 = 2uv$.	From $A2$ by algebra.
$A4 : u^2 + v^2 = w^2$.	Pythagorean theorem.
$A5 : u^2 + v^2 = 2uv$	Substituting w^2 from $A3$ in $A4$.
$A6 : u^2 - 2uv + v^2 = 0$.	From $A5$ by algebra.
$A7 : u - v = 0$.	Factoring $A6$ and taking square root.

Thus, $u = v$, completing the proof.

Proof. Because $\sin(U) = \sqrt{u/2v}$ and also $\sin(U) = u/w$, $\sqrt{u/2v} = u/w$, or, $w^2 = 2uv$. Now from the Pythagorean theorem, $w^2 = u^2+v^2$. On substituting $2uv$ for w^2 and then performing algebraic manipulations, one has $u = v$. $\square$

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4.8 **Analysis of Proof.** The appearance of the key words "there is" in the conclusion suggests using the construction method to find a real number x such that $x^2 - 5x/2 + 3/2 = 0$. Factoring this equation means you want to find a real number x such that $(x - 3/2)(x - 1) = 0$. So the desired real number is either $x = 1$ or $x = 3/2$ which, when substituted in $x^2 - 5x/2 + 3/2$, yields 0. The real number is not unique as either $x = 1$ or $x = 3/2$ works.

Proof. Factoring $x^2 - 5x/2 + 3/2 = 0$ yields $(x - 3/2)(x - 1) = 0$, so $x = 1$ or $x = 3/2$. Thus, there exists a real number, namely, $x = 1$ or $x = 3/2$, such that $x^2 - 5x/2 + 3/2 = 0$. The real number is not unique. $\square$

4.15 The proof is not correct. The mistake occurs because the author uses the same symbol x for the element that is in $R \cap S$ and in $S \cap T$ where, in fact, the element in $R \cap S$ need not be the same as the element in $S \cap T$.

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Web Solutions to Exercises

5.3 a. $\exists$ a mountain $\ni$ $\forall$ other mountains, this one is taller than the others.
b. $\forall$ angle t, $\sin(2t) = 2\sin(t)\cos(t)$.
c. $\forall$ nonnegative real numbers p and q, $\sqrt{pq} \geq (p+q)/2$.
d. $\forall$ real numbers x and y with $x < y$, $\exists$ a rational number $r \ni \ x < r < y$.

5.9 Key Question: How can I show that a real number (namely, v) is an upper bound for a set of real numbers (namely, S)?
Key Answer: Show that every element in the set is $\leq$ the number and so it must be shown that
$B1$: For every element $s \in S$, $s \leq v$.
$A1$: Choose an element $s' \in S$ for which it must be shown that
$B2$: $s' \leq v$.

5.18 **Analysis of Proof.** The appearance of the quantifier "for every" in the conclusion suggests using the choose method, whereby one chooses

$A1$: An element $t \in T$,

for which it must be shown that

$B1$: t is an upper bound for the set S.

A key question associated with $B1$ is, "How can I show that a real number (namely, t) is an upper bound for a set (namely, S)?" By definition, one must show that

$B2:$ For every element $x \in S$, $x \leq t$.

The appearance of the quantifier "for every" in the backward statement $B2$ suggests using the choose method, whereby one chooses

$A2:$ An element $x \in S$,

for which it must be shown that

$B3:$ $x \leq t$.

To do so, work forward from $A2$ and the definition of the set S in the hypothesis to obtain

$A3:$ $x(x-3) \leq 0$.

From $A3$, either $x \geq 0$ and $x - 3 \leq 0$, or, $x \leq 0$ and $x - 3 \geq 0$. But the latter cannot happen, so

$A4:$ $x \geq 0$ and $x - 3 \leq 0$.

From $A4$,

$A5:$ $x \leq 3$.

But, from $A1$ and the definition of the set T in the hypothesis

$A6:$ $t \geq 3$.

Combining $A5$ and $A6$ yields $B3$, thus completing the proof.

5.22 **Analysis of Proof.** The forward-backward method gives rise to the key question, "How can I show that a set (namely, C) is convex?" One answer is by the definition, whereby it must be shown that

$B1:$ For all elements x and y in C, and for all real numbers t with $0 \leq t \leq 1, tx + (1-t)y \in C$.

The appearance of the quantifiers "for all" in the backward statement $B1$ suggests using the choose method to choose

$A1:$ Elements x and y in C, and a real number t with $0 \leq t \leq 1$,

for which it must be shown that

$B2:$ $tx + (1-t)y \in C$, that is, $a[tx + (1-t)y] \leq b$.

Turning to the forward process, because x and y are in C (see $A1$),

$A2:$ $ax \leq b$ and $ay \leq b$.

Multiplying both sides of the two inequalities in $A2$, respectively, by the nonnegative numbers t and $1-t$ (see $A1$) and adding the inequalities yields:

$$A3: \; tax + (1-t)ay \leq tb + (1-t)b.$$

Performing algebra on $A3$ yields $B2$, and so the proof is complete the proof.

Proof. Let t be a real number with $0 \leq t \leq 1$, and let x and y be in C. Then $ax \leq b$ and $ay \leq b$. Multiplying both sides of these inequalities by $t \geq 0$ and $1-t \geq 0$, respectively, and adding yields $a[tx + (1-t)y] \leq b$. Hence, $tx + (1-t)y \in C$. Therefore, C is a convex set and the proof is complete. □

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Web Solutions to Exercises

6.1 The reason you need to show that Y has the certain property is because you only know that the something happens for objects with the certain property. You do not know that the something happens for objects that do not satisfy the certain property. Therefore, if you want to use specialization to claim that the something happens for this particular object Y, you must be sure that Y has the certain property.

6.16 **Analysis of Proof.** The forward-backward method gives rise to the key question, "How can I show that a function (namely, f) is $\geq$ another function (namely, h) on a set (namely, S)?" According to the definition, it is necessary to show that

$B1:$ For every element $x \in S$, $f(x) \geq h(x)$.

Recognizing the key words "for all" in the backward statement $B1$, the choose method is used to choose

$A1:$ An element $x \in S$,

for which it must be shown that

$B2:$ $f(x) \geq h(x)$.

Turning now to the forward process, because $f \geq g$ on S, by definition,

$A2:$ For every element $y \in S$, $f(y) \geq g(y)$.

(Note the use of the symbol y so as not to overlap with the symbol x in $A1$.) Likewise, because $g \geq h$ on S, by definition,

$A3$: For every element $z \in S$, $g(z) \geq h(z)$.

Recognizing the key words "for every" in the forward statements $A2$ and $A3$, the desired conclusion in $B2$ is obtained by specialization. Specifically, specializing $A2$ with $y = x$ (noting that $x \in S$ from $A1$) yields:

$A4$: $f(x) \geq g(x)$.

Likewise, specializing $A3$ with $z = x$ (noting that $x \in S$ from $A1$) yields:

$A5$: $g(x) \geq h(x)$.

Combining $A4$ and $A5$, you have

$A6$: $f(x) \geq g(x) \geq h(x)$.

The proof is now complete because $A6$ is the same as $B2$.

Proof. Let $x \in S$. From the hypothesis that $f \geq g$ on S, it follows that $f(x) \geq g(x)$. Likewise, because $g \geq h$ on S, you have $g(x) \geq h(x)$. Combining these two means that $f(x) \geq g(x) \geq h(x)$ and so $f \geq h$ on S. □

6.20 **Analysis of Proof.** The appearance of the quantifier "for all" in the conclusion indicates that you should use the choose method to choose

$A1$: A real number $s' \geq 0$,

for which it must be shown that

$B1$: The function $s'f$ is convex.

An associated key question is, "How can I show that a function (namely, $s'f$) is convex?" Using the definition in Exercise 5.2(c), one answer is to show that

$B2$: For all real numbers x and y, and for all t with $0 \leq t \leq 1$,
$s'f(tx + (1-t)y) \leq ts'f(x) + (1-t)s'f(y)$.

The appearance of the quantifier "for all" in the backward process suggests using the choose method to choose

$A2$: Real numbers x' and y', and $0 \leq t' \leq 1$,

for which it must be shown that

$B3$: $s'f(t'x' + (1-t')y') \leq t's'f(x') + (1-t')s'f(y')$.

The desired result is obtained by working forward from the hypothesis that f is a convex function. By the definition in Exercise 5.2(c), you have that

$A3:$ For all real numbers x and y, and for all t with $0 \le t \le 1$,
$f(tx + (1-t)y) \le tf(x) + (1-t)f(y)$.

Specializing the statement in $A3$ to $x = x'$, $y = y'$, and $t = t'$ (noting that $0 \le t' \le 1$) yields

$A4:$ $f(t'x' + (1-t')y') \le t'f(x') + (1-t')f(y')$.

The desired statement $B3$ is obtained by multiplying both sides of the inequality in $A4$ by the nonnegative number s', thus completing the proof.

Proof. Let $s' \ge 0$. To show that $s'f$ is convex, let x' and y' be real numbers, and let t' with $0 \le t' \le 1$. It will be shown that $s'f(t'x' + (1-t')y') \le t's'f(x') + (1-t')s'f(y')$.

Because f is a convex function by hypothesis, it follows from the definition that $f(t'x'+(1-t')y') \le t'f(x')+(1-t')f(y')$. The desired result is obtained by multiplying both sides of this inequality by the nonnegative number s'. $\square$

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Web Solutions to Exercises

7.20 **Analysis of Proof.** The forward-backward method gives rise to the key question, "How can I show that a function (namely, f) is bounded above?" One answer is by the definition, whereby one must show that

$B1$: There is a real number y such that for every real number x, $-x^2 + 2x \leq y$.

The appearance of the quantifier "there is" in the backward statement $B1$ suggests using the construction method to produce the desired value for y. Trial and error might lead you to construct $y = 1$ (any value of $y \geq 1$ will also work). Now it must be shown that this value of $y = 1$ is correct, that is:

$B2$: For every real number x, $-x^2 + 2x \leq 1$.

The appearance of the quantifier "for all" in the backward statement $B2$ suggests using the choose method to choose

$A1$: A real number x,

for which it must be shown that

$B3$: $-x^2 + 2x \leq 1$, that is, $x^2 - 2x + 1 \geq 0$.

But because $x^2 - 2x + 1 = (x - 1)^2$, this number is always ≥ 0. Thus $B3$ is true, completing the proof.

Proof. To see that the function $f(x) = -x^2 + 2x$ is bounded above, it will be shown that for all real numbers x, $-x^2 + 2x \leq 1$. To that end, let x be any real number. Then $x^2 - 2x + 1 = (x-1)^2 \geq 0$, thus completing the proof. □

7.22 **Analysis of Proof.** The first key words in the conclusion from the left are "for every," so the choose method is used to choose

$A1$: A real number $\epsilon > 0$,

for which it must be shown that

$B1$: There is an element $x \in S$ such that $x > 1 - \epsilon$.

Recognizing the key words "there is" in the backward statement $B1$, the construction method is used to produce the desired element in S. To that end, from the hint, you can write S as follows:

$$S = \{\text{real numbers } x : \text{there is an integer } n \geq 2 \text{ with } x = 1 - 1/n\}.$$

Thus, you can construct $x = 1 - 1/n$, for an appropriate choice of the integer $n \geq 2$. To find the value for n, from $B1$, you want

$$x = 1 - 1/n > 1 - \epsilon, \quad \text{that is,} \quad 1/n < \epsilon, \quad \text{that is,} \quad n > 1/\epsilon.$$

In summary, noting that $\epsilon > 0$ from $A1$, if you let $n \geq 2$ be any integer $> 1/\epsilon$, then $x = 1 - 1/n \in S$ satisfies the desired property in $B1$, namely, that $x = 1 - 1/n > 1 - \epsilon$. The proof is now complete.

Proof. Let $\epsilon > 0$. To see that there is an element $x \in S$ such that $x > 1 - \epsilon$, let $n \geq 2$ be an integer for which $n > 1/\epsilon$. It then follows from the defining property that $x = 1 - 1/n \in S$ and by the choice of n that $x = 1 - 1/n > 1 - \epsilon$. It has thus been shown that for every real number $\epsilon > 0$, there is an element $x \in S$ such that $x > 1 - \epsilon$, thus completing the proof. □

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Web Solutions to Exercises

THERE ARE NO APPLICABLE EXERCISES FOR CHAPTER 8.

9

Web Solutions to Exercises

9.12 **Analysis of Proof.** By contradiction, assume A and $NOT\ B$, that is:

$$\begin{aligned} A:&\quad n-1,\ n, \text{ and } n+1 \text{ are consecutive positive integers.} \\ A1\ (NOT\ B):&\quad (n+1)^3 = n^3 + (n-1)^3. \end{aligned}$$

A contradiction is reached by showing that

$$B1: \ n^2(n-6) = 2 \text{ and } n^2(n-6) \geq 49.$$

To that end, rewriting $A1$ by algebra, you have:

$$\begin{aligned} A2:&\quad n^3 + 3n^2 + 3n + 1 = n^3 + n^3 - 3n^2 + 3n - 1, \text{ or} \\ A3:&\quad n^3 - 6n^2 = 2, \text{ or} \\ A4:&\quad n^2(n-6) = 2. \end{aligned}$$

From $A4$, because $n^2 > 0$, it must be that

$$A5: \ n - 6 > 0, \text{ that is, } n \geq 7.$$

But when $n \geq 7$, $n - 6 \geq 1$, and $n^2 \geq 49$, so

$$A6: \ n^2(n-6) \geq n^2 \geq 49.$$

Now $A6$ contradicts $A4$ because $A6$ states that $n^2(n-6) \geq 49$ while $A4$ states that $n^2(n-6) = 2$. This contradiction completes the proof.

Proof. Assume, to the contrary, that the three consecutive integers $n-1$, n, and $n+1$ satisfy

$$(n+1)^3 = n^3 + (n-1)^3.$$

Expanding these expressions and rewriting yields

$$n^2(n-6) = 2.$$

Because $n^2 > 0$, $n - 6 > 0$, that is, $n \geq 7$. But then $n^2(n-6) \geq n^2 \geq 49$. This contradicts the fact that $n^2(n-6) = 2$, thus completing the proof. □

9.17 **Analysis of Proof.** When using the contradiction method, you can assume the hypothesis that

$A:\ x \geq 0,\ y \geq 0,\ x + y = 0,$

and also that

$A1$ ($NOT\ B$): Either $x \neq 0$ or $y \neq 0$.

From $A1$, suppose first that

$A2:\ x \neq 0.$

Because $x \geq 0$ from A, it must be that

$A3:\ x > 0.$

A contradiction to the fact that $y \geq 0$ is reached by showing that

$B1:\ y < 0.$

Specifically, because $x + y = 0$ from A,

$A4:\ y = -x.$

Because $-x < 0$ from $A3$, a contradiction has been reached. A similar argument applies for the case where $y \neq 0$ (see $A1$).

Proof. Assume that $x \geq 0$, $y \geq 0$, $x + y = 0$, and that either $x \neq 0$ or $y \neq 0$. If $x \neq 0$, then $x > 0$ and $y = -x < 0$, but this contradicts the fact that $y \geq 0$. Similarly, if $y \neq 0$, then $y > 0$, and $x = -y < 0$, but this contradicts the fact that $x \geq 0$. □

9.24 **Analysis of Proof.** By the contradiction method, assume that

$A1:$ The number of primes is finite.

From $A1$, there will be a prime number that is larger than all the other prime numbers. So,

$A2:$ Let n be the largest prime number.

Consider the number $n! + 1$ and let

$A3:$ p be any prime number that divides $n!+1$.

A contradiction is reached by showing that

$B1:$ $p > n$.

To see that $p > n$, by contradiction, suppose that $1 < p \leq n$. From the proposition in Exercise 9.23, it follows that p does not divide $n!+1$, which contradicts $A3$ and completes the proof.

Proof. Assume, to the contrary, that there are a finite number of primes. Let n be the largest prime and let p be any prime divisor of $n!+1$. Now if $1 < p \leq n$, then by the proposition in Exercise 9.23, it follows that p does not divide $n!+1$. This contradiction completes the proof. □

9.27 **Analysis of Proof.** The proof is by contradiction, so the author assumes that

$A1$ (NOT B): The polynomial x^4+2x^2+2x+2 can be expressed as the product of the two polynomials x^2+ax+b and x^2+cx+d in which a, b, c, and d are integers.

Working forward by multiplying the two polynomials, you have that

$$A2: \; x^4+2x^2+2x+2 = (x^2+ax+b)(x^2+cx+d) = \\ x^4+(a+c)x^3+(b+ac+d)x^2+(bc+ad)x+bd.$$

Equating coefficients of like powers of x on both sides, it follows that

$A3:$ $a+c=0$.
$A4:$ $b+ac+d=2$.
$A5:$ $bc+ad=2$.
$A6:$ $bd=2$.

From $A6$, b is odd (± 1) and d is even (± 2) or vice versa. Suppose, first, that

$A7:$ **Case 1:** b is odd and d is even.

(Subsequently, the case when b is even and d is odd is considered.) It now follows that, because the right side of $A5$ is even, the left side is also. Because d is even, so is ad. It must therefore be the case that

$A8:$ bc is even.

However, from $A7$, b is odd, so it must be that

$A9:$ c is even.

But then, from the left side of $A4$, you have

$A10:$ $b+ac+d$ is odd + even + even, which is odd.

This is because b is odd (see $A7$), ac is even (from $A9$), and d is even (see $A7$). However, $A10$ is a contradiction because the right side of $A4$, namely, 2, is even. This establishes a contradiction for Case 1 in $A7$. A similar contradiction is reached in Case 2, when b is even and d is odd, thus completing the proof.

10

Web Solutions to Exercises

10.17 **Analysis of Proof.** With the contrapositive method, you assume

$A1$ ($NOT\ B$): There is an integer solution, say m, to the equation $n^2 + n - c = 0$.

You must then show that

$B1$ ($NOT\ A$): c is not odd, that is, c is even.

But from $A1$, you have that

$A2$: $c = m + m^2$.

Observe that $m + m^2 = m(m + 1)$ is the product of two consecutive integers and is therefore even, thus establishing $B1$ and completing the proof.

Proof. Assume that there is an integer solution, say m, to the equation $n^2 + n - c = 0$. It will be shown that c is even. But $c = m + m^2 = m(m + 1)$ is even because the product of two consecutive integers is even. $\square$

10.20 **Analysis of Proof.** By the contrapositive method, you can assume

$A1$ ($NOT\ B$): The quadrilateral $RSTU$ is not a rectangle.

You must then show that

$B1$ ($NOT\ A$): There is an obtuse angle.

The appearance of the quantifier "there is" in the backward statement $B1$ suggests turning to the forward process to construct the obtuse angle.

Working forward from $A1$, you can conclude that

$A2$: At least one angle of the quadrilateral is not 90 degrees, say angle R.

If angle R has more than 90 degrees, then R is the desired angle and the proof is complete. Otherwise,

$A3$: Angle R has less than 90 degrees.

Because the sum of all the angles in $RSTU$ is 360 degrees, $A3$ means that

$A4$: The remaining angles of the quadrilateral must add up to more than 270 degrees.

Among these three angles that add up to more than 270 degrees, one of them must be greater than 90 degrees, and that is the desired obtuse angle. The proof is now complete.

Proof. Assume that the quadrilateral $RSTU$ is not a rectangle, and hence, one of its angles, say R, is not 90 degrees. An obtuse angle will be found. If angle R has more than 90 degrees, then R is the desired obtuse angle. Otherwise the remaining three angles add up to more than 270 degrees. Thus one of the remaining three angles is obtuse, and so the proof is complete. □

10.22 **Analysis of Proof.** By the contrapositive method, you can assume

$A1$ ($NOT\ B$): $x < 0$.

It must be shown that

$B1$ ($NOT\ A$): There is a real number $\epsilon > 0$ such that $x < -\epsilon$.

Recognizing the key words "there is" in the backward statement $B1$, the construction method is used to produce the desired $\epsilon > 0$. Turning to the forward process to do so, from $A1$, because $x < 0$, construct ϵ as any value with $0 < \epsilon < -x$. (Note that this construction is possible because $-x > 0$.) By design, $\epsilon > 0$ and, because $\epsilon < -x$, it follows that $x < -\epsilon$. Thus ϵ has all the needed properties in $B1$, and the proof is complete.

Proof. Assume, to the contrary, that $x < 0$. It will be shown that there is a real number $\epsilon > 0$ such that $x < -\epsilon$. To that end, construct ϵ as any value with $0 < \epsilon < -x$ (noting that this is possible because $-x > 0$). Clearly $\epsilon > 0$ and because $\epsilon < -x$, $x < -\epsilon$, thus completing the proof. □

11

Web Solutions to Exercises

11.10 **Analysis of Proof.** According to the indirect uniqueness method, one must first construct a real number x for which $mx + b = 0$. But because the hypothesis states the $m \neq 0$, you can construct

$A1:\ x = -b/m.$

This value is correct because

$A2:\ mx + b = m(-b/m) + b = -b + b = 0.$

To establish the uniqueness by the indirect uniqueness method, suppose that

$A3:$ y is a real number with $y \neq x$ such that $my + b = 0.$

A contradiction to the hypothesis that $m \neq 0$ is reached by showing that

$B1:\ m = 0.$

Specifically, from $A2$ and $A3$,

$A4:\ mx + b = my + b.$

Subtracting the right side of the equality in $A4$ from the left side and rewriting yields

$A5:\ m(x - y) = 0.$

On dividing both sides of the equality in $A5$ by the nonzero number $x - y$ (see $A3$), it follows that $m = 0$. This contradiction establishes the uniqueness.

Proof. To construct the number x for which $mx + b = 0$, let $x = -b/m$ (because $m \neq 0$). Then $mx + b = m(-b/m) + b = 0$.

Now suppose that $y \neq x$ and also that $my + b = 0$. Then $mx + b = my + b$, and so $m(x - y) = 0$. But because $x - y \neq 0$, it must be that $m = 0$. This contradicts the hypothesis that $m \neq 0$ and completes the proof. $\square$

12

Web Solutions to Exercises

12.3 a. The time to use induction instead of the choose method to show that, "For every integer $n \geq n_0$, $P(n)$ is true" is when you can relate $P(n)$ to $P(n-1)$, for then you can use the induction hypothesis that $P(n-1)$ is true, and this should help you establish that $P(n)$ is true. If you were to use the choose method, you would choose

$A1:$ An integer $n \geq n_0$,

for which it must be shown that

$B1:$ $P(n)$ is true.

With the choose method, you cannot use the assumption that $P(n-1)$ is true to do so.

b. It is not possible to use induction when the object is a real number because showing that $P(n)$ implies $P(n+1)$ "skips over" many values of the object. As a result, the statement will not have been proved for such values.

12.9 **Proof.** First it is shown that $P(n)$ is true for $n = 5$. But $2^5 = 32$ and $5^2 = 25$, so $2^5 > 5^2$ and so $P(n)$ is true for $n = 5$. Assuming that $P(n)$ is true, you must then prove that $P(n+1)$ is true. So assume

$$P(n): \; 2^n > n^2.$$

It must be shown that

$P(n+1): \ 2^{n+1} > (n+1)^2.$

Starting with the left side of $P(n+1)$ and using the fact that $P(n)$ is true, you have:

$$2^{n+1} = 2(2^n) > 2(n^2).$$

To obtain $P(n+1)$, it must still be shown that for $n > 5$, $2n^2 > (n+1)^2 = n^2+2n+1$, or, by subtracting n^2+2n-1 from both sides and factoring, that $(n-1)^2 > 2$. This last statement is true because, for $n > 5$, $(n-1)^2 \geq 4^2 = 16 > 2$. □

12.17 **Proof.** For $n = 1$ the statement becomes:

$P(1): \ [\cos(x) + i\sin(x)]^1 = \cos(1x) + i\sin(1x).$

Now $P(1)$ is true because both sides evaluate to $\cos(x) + i\sin(x)$.

Now assume the statement is true for $n-1$, that is:

$P(n-1): \ [\cos(x) + i\sin(x)]^{n-1} = \cos((n-1)x) + i\sin((n-1)x).$

It must be shown that $P(n)$ is true, that is:

$P(n): \ [\cos(x) + i\sin(x)]^n = \cos(nx) + i\sin(nx).$

Using $P(n-1)$ and the facts that

$$\begin{array}{rcl} \cos(a+b) & = & \cos(a)\cos(b) - \sin(a)\sin(b), \\ \sin(a+b) & = & \sin(a)\cos(b) + \cos(a)\sin(b), \end{array}$$

starting with the left side of $P(n)$, you have:

$$\begin{array}{rcl} [\cos(x) + i\sin(x)]^n & = & [\cos(x) + i\sin(x)]^{n-1}[\cos(x) + i\sin(x)] \\ & = & [\cos((n-1)x) + i\sin((n-1)x)][\cos(x) + i\sin(x)] \\ & = & [\cos((n-1)x)\cos(x) - \sin((n-1)x)\sin(x)] + \\ & & i[\sin((n-1)x)\cos(x) + \cos((n-1)x)\sin(x)] \\ & = & \cos(nx) + i\sin(nx). \end{array}$$

This establishes that $P(n)$ is true, thus completing the proof. □

12.20 The author relates $P(n+1)$ to $P(n)$ by using the product rule of differentiation to express

$$[x(x^n)]' = (x)'(x^n) + x(x^n)'.$$

The author then uses the induction hypothesis to replace $(x^n)'$ with nx^{n-1}.

12.25 The proof is incorrect because when $n = 1$ and $r = 1$, the right side of $P(1)$ is undefined because you cannot divide by zero. A similar problem arises throughout the rest of the proof.

13

Web Solutions to Exercises

13.6 The author uses a proof by cases when the following statement containing the key words "either/or" is encountered in the forward process:

$A1$: Either the factor b is odd or even.

Accordingly, the author considers the following two cases:

Case 1: The factor b is odd. The author then works forward from this information to establish the contradiction that the left side of equation (2) is odd and yet is equal to the even number 2.

Case 2: The factor b is even. The author claims, without providing details, that this case also leads to a contradiction.

13.11 **Analysis of Proof.** With this proof by elimination, you assume that

$$\begin{aligned} A: &\quad x^3 + 3x^2 - 9x - 27 \geq 0 \text{ and} \\ A1\ (NOT\ D): &\quad x < 3. \end{aligned}$$

It must be shown that

$B1$ (C): $x \leq -3$.

By factoring A, it follows that

$A2$: $x^3 + 3x^2 - 9x - 27 = (x-3)(x+3)^2 \geq 0$.

Dividing both sides of $A2$ by $x - 3 < 0$ (from $A1$) yields

$A3: \ (x+3)^2 \le 0.$

Because $(x+3)^2$ is also ≥ 0, from $A3$, it must be that

$A4: \ (x+3)^2 = 0$, so $x+3=0$, that is, $x=-3$.

Thus $B1$ is true, completing the proof.

Proof. Assume that $x^3+3x^2-9x-27 \ge 0$ and $x < 3$. Then it follows that $x^3+3x^2-9x-27 = (x-3)(x+3)^2 \ge 0$. Because $x < 3$, $(x+3)^2$ must be 0, so $x+3=0$, that is, $x=-3$. Thus, $x \le -3$, completing the proof. □

13.12 **Analysis of Proof.** Observe that the conclusion can be written as:

$B: $ Either $a=b$ or $a=-b$.

The key words "either/or" in the backward process now suggest proceeding with a proof by elimination, in which you can assume the hypothesis and

$A1: \ a \ne b.$

It must be shown that

$B1: \ a = -b.$

Working forward from the hypotheses that $a|b$ and $b|a$, by definition:

$A2:$ There is an integer k such that $b=ka$.
$A3:$ There is an integer m such that $a=mb$.

Substituting $a=mb$ in the equality in $A2$ yields:

$A4: \ b = kmb.$

If $b=0$ then, from $A3$, $a=0$ and so $B1$ is clearly true and the proof is complete. Thus, you can assume that $b \ne 0$. Therefore, on dividing both sides of the equality in $A4$ by b you obtain:

$A5: \ km = 1.$

From $A5$ and the fact that k and m are integers (see $A2$ and $A3$), it must be that

$A6:$ Either $(k=1$ and $m=1)$ or $(k=-1$ and $m=-1)$.

Recognizing the key words "either/or" in the forward statement $A6$, a proof by cases is used.

Case 1: $k=1$ and $m=1$. In this case, $A2$ leads to $a=b$, which cannot happen according to $A1$.

Case 2: $k=-1$ and $m=-1$. In this case, from $A2$, it follows that $a=-b$, which is precisely $B1$, thus completing the proof.

Proof. To see that $a = \pm b$, assume that $a|b$, $b|a$, and $a \neq b$. It will be shown that $a = -b$. By definition, it follows that there are integers k and m such that $b = ka$ and $a = mb$. Consequently, $b = kmb$. If $b = 0$, then $a = mb = 0$ and so $a = -b$. Thus, assume that $b \neq 0$. It then follows that $km = 1$. Because k and m are integers, it must be that $k = m = 1$ or $k = m = -1$. However, because $a \neq b$, it must be that $k = m = -1$. From this it follows that $a = mb = -b$, and so the proof is complete. $\square$

14 Web Solutions to Exercises

14.7 **Analysis of Proof.** Recognizing the key word "min" in the conclusion and letting $z = \max\{ub : ua \leq c, u \geq 0\}$, the max/min methods results in the following equivalent quantified statement:

$B1:$ For every real number x with $ax \geq b$ and $x \geq 0$, it follows that $cx \geq z = \max\{ub : ua \leq c, u \geq 0\}$.

The key words "for every" in the backward statement $B1$ suggest using the choose method to choose

$A1:$ A real number x with $ax \geq b$ and $x \geq 0$,

for which it must be shown that

$B2:$ $cx \geq \max\{ub : ua \leq c, u \geq 0\}$.

Recognizing the key word "max" in $B2$, the max/min methods lead to the need to prove the following equivalent quantified statement:

$B3:$ For every real number u with $ua \leq c$ and $u \geq 0$, $cx \geq ub$.

The key words "for every" in the backward statement $B3$ suggest using the choose method to choose

$A2:$ A real number u with $ua \leq c$ and $u \geq 0$,

for which it must be shown that

$B4:$ $cx \geq ub$.

To reach $B4$, multiply both sides of $ax \geq b$ in $A1$ by $u \geq 0$ to obtain

$$A3: \ uax \geq ub.$$

Likewise, multiply both sides of $ua \leq c$ in $A2$ by $x \geq 0$ to obtain

$$A4: \ uax \leq cx.$$

The desired conclusion in $B4$ that $cx \geq ub$ follows by combining $A3$ and $A4$.

Proof. To reach the desired conclusion that $cx \geq ub$, let x and u be real numbers with $ax \geq b$, $x \geq 0$, $ua \leq c$, and $u \geq 0$. Multiplying $ax \geq b$ through by $u \geq 0$ and $ua \leq c$ through by $x \geq 0$, it follows that $ub \leq uax \leq cx$. $\square$

15

Web Solutions to Exercises

15.5 a. Using induction, you would first have to show that $4! > 4^2$. Then you would assume that $n! > n^2$ and $n \geq 4$, and show that $(n+1)! > (n+1)^2$.

b. Using the choose method, you would choose an integer m for which $m \geq 4$. You would then try to show that $m! > m^2$.

c. Converting the statement to the form, "If ... then ..." you obtain, "If n is an integer with $n \geq 4$, then $n! > n^2$." With the forward-backward method, you would then assume that n is an integer with $n \geq 4$ and work forward to show that $n! > n^2$.

d. Using contradiction, you would assume that there is an integer $n \geq 4$ such that $n! \leq n^2$ and then work forward to reach a contradiction.

Appendix A Web Solutions to Exercises

A.2 One different key question is, “How can I show that the intersection of two sets is a subset of another set.” (Other answers are possible.)

A.8 **Analysis of Proof.** The first step of induction is to show that the statement is true for $n = 1$ which, for this exercise, means you must show that the following statement is true:

> **P(1):** If the sets S and T each have $n = 1$ elements, then $S \times T$ has $1^2 = 1$ element.

Now $P(1)$ is true because if $S = \{s\}$ and $T = \{t\}$, then $S \times T = \{(s, t)\}$, which has exactly one element.

The next step of induction is to assume that the statement is true for n, that is, assume that

> **P(n):** If the sets S and T each have n elements, then $S \times T$ has n^2 elements.

You must then show that the statement is true for $n+1$, that is, you must show that

> **P(n+1):** If the sets S and T each have $n+1$ elements, then $S \times T$ has $(n+1)^2 = n^2 + 2n + 1$ elements.

The key to a proof by induction is to write $P(n+1)$ in terms of $P(n)$. To that end, suppose that

$$S = \{s_1, \ldots, s_n, s_{n+1}\} \quad \text{and} \quad T = \{t_1, \ldots, t_n, t_{n+1}\}.$$

Letting S' and T' be the sets consisting of the first n elements of S and T, respectively, from the induction hypothesis you know that $S' \times T'$ has n^2 elements. It remains to show that $S \times T$ contains $2n+1$ elements in addition to the n^2 elements of $S' \times T'$. But, the remaining elements of $S \times T$ consist of the following. For the element $s_{n+1} \in S$, you can construct the following additional n elements of $S \times T$: $(s_{n+1}, t_1), \ldots, (s_{n+1}, t_n)$. Likewise, for the element $t_{n+1} \in T$, you can construct the following additional n elements of $S \times T$: $(s_1, t_{n+1}), \ldots, (s_n, t_{n+1})$. The final element of $S \times T$ is (s_{n+1}, t_{n+1}). The proof is now complete.

Proof. If the sets S and T each have $n = 1$ elements, then $S \times T$ has $1^2 = 1$ element because if $S = \{s\}$ and $T = \{t\}$, then $S \times T = \{(s, t)\}$, which has exactly one element.

Assume now that the statement is true for n. It must be shown that if the sets S and T each have $n+1$ elements, then $S \times T$ has $(n+1)^2 = n^2 + 2n + 1$ elements. To that end, letting S' and T' be the sets consisting of the first n elements of S and T, respectively, you can count the following number of elements in $S \times T$:

$S' \times T'$	n^2 elements by the induction hypothesis
$(s_{n+1}, t_1), \ldots, (s_{n+1}, t_n)$	n elements
$(s_1, t_{n+1}), \ldots, (s_n, t_{n+1})$	n elements
(s_{n+1}, t_{n+1})	1 element

Putting together the pieces, $S \times T$ has $n^2 + n + n + 1 = (n+1)^2$ elements, and so the proof is complete. □

A.14 The desired condition for $f(x) = ax + b$ to be injective is that $a \neq 0$, as established in the following proof.

Analysis of Proof. A key question associated with the conclusion is, "How can I show that a function (namely, $f(x) = ax + b$) is injective?" Using the definition, one answer is to show that

> **B1:** For all real numbers u and v with $u \neq v$, $au + b \neq av + b$.

Recognizing the key words "for all" in the backward statement $B1$, you should now use the choose method to choose

A1: Real numbers u and v with $u \neq v$,

for which you must show that

B2: $au + b \neq av + b$.

Recognizing the key word "not" in $B2$ (and also in $A1$), you should now consider using either the contradiction or contrapositive method. Here, the contrapositive method is used. So, you can assume that

A2 (NOT B2): $au + b = av + b$.

According to the contrapositive method, you must now show that

B3 (NOT A1): $u = v$.

You can obtain $B3$ by working forward from $A2$ and the assumption that $a \neq 0$, as follows:

$$\begin{array}{cl} au + b = av + b & \text{(from } A2\text{)} \\ au = av & \text{(subtract } b \text{ from both sides)} \\ u = v & \text{(divide both sides by } a \neq 0\text{).} \end{array}$$

The proof is now complete.

Proof. To show that the function $f(x) = ax + b$ is injective, assuming that $a \neq 0$, let u and v be real numbers with $f(u) = f(v)$, that is, $au + b = av + b$. It will be shown that $u = v$, but this is true because

$$\begin{array}{cl} au + b = av + b & \text{(from } A2\text{)} \\ au = av & \text{(subtract } b \text{ from both sides)} \\ u = v & \text{(divide both sides by } a \neq 0\text{).} \end{array}$$

The proof is now complete. □

Appendix B Web Solutions to Exercises

B.10 a. A common key question associated with the properties in Table B.2 is, “How can I show that two matrices are equal?”

b. Using the definition, one answer to the key question in part (a) is to show that the matrices have the same dimensions and that all elements of the two matrices are equal. Applying this answer to the matrices $A+B = B+A$ that both have the dimensions $m \times n$ means you must show that for every integer $i = 1, \ldots, m$ and $j = 1, \ldots, n$, $(A+B)_{ij} = (B+A)_{ij}$, that is, you must show that

B1: For every integer $i = 1, \ldots, m$ and for every integer $j = 1, \ldots, n$, $A_{ij} + B_{ij} = B_{ij} + A_{ij}$.

c. Based on the answer in part (b), the next technique you should use is the choose method because the key words “for every” appear in the backward statement $B1$. For that statement, you would choose

A1: Integers i and j with $1 \leq i \leq m$ and $1 \leq j \leq n$,

for which you must show that

B2: $A_{ij} + B_{ij} = B_{ij} + A_{ij}$.

Appendix C
Web Solutions to Exercises

C.3 **Analysis of Proof.** The forward-backward method is used to begin the proof because there are no key words in the hypothesis or conclusion. A key question associated with the conclusion $a|(b+c)$ is, "How can I show that an integer (namely, a) divides another integer (namely, $b+c$)?" Using the definition, one answer is to show that

B1: There is an integer d such that $b + c = da$.

Recognizing the key words "there is" in the backward statement $B1$, you should now use the construction method. Specifically, turn to the forward process to construct the desired integer d.

Working forward from the hypothesis that a divides b, by definition, you know that

A1: There is an integer e such that $b = ea$.

Likewise, from the hypothesis that a divides c, by definition,

A2: There is an integer f such that $c = fa$.

Adding the equalities in $A1$ and $A2$ results in

A3: $b + c = ea + fa = (e + f)a$.

From $A3$ you can see that the desired value of the integer d in $B1$ is $d = e+f$. According to the construction method, you must still show that this value of d satisfies the certain property and the something that happens in $B1$, namely, that $b + c = da$, but this is clear from $A3$.

Proof. To show that $a|(b + c)$, by definition, it is shown that there is an integer d such that $b + c = da$. However, from the hypothesis that $a|b$, there is an integer e such that $b = ea$. Likewise, from the hypothesis that $a|c$, there is an integer f such that $c = fa$. Letting $d = e + f$, it is easy to see that $b + c = ea + fa = (e + f)a = da$, thus completing the proof. □

C.8 **Analysis of Proof.** To reach the conclusion, work forward from the hypothesis that $a \odot b = a \odot c$ by combining both sides on the left with the element a^{-1} (which exists by property (3) of a group) to obtain

A1: $a^{-1} \odot (a \odot b) = a^{-1} \odot (a \odot c)$.

Now specialize the for-all statement in property (1) of a group to both sides of $A1$ to obtain

A2: $(a^{-1} \odot a) \odot b = (a^{-1} \odot a) \odot c$.

Using the fact that $a^{-1} \odot a = e$ from property (3) of a group, $A2$ becomes

A3: $e \odot b = e \odot c$.

Finally, specialize property (2) of a group to both sides of $A3$ to obtain

A4: $b = c$.

The proof is now complete because $A4$ is the same as the conclusion of the proposition. □

Proof. To reach the conclusion that $b = c$, you have that

$$\begin{array}{rcll}
a \odot b & = & a \odot c & \text{(hypothesis)} \\
a^{-1} \odot (a \odot b) & = & a^{-1} \odot (a \odot c) & \text{(combine both sides with } a^{-1}\text{)} \\
(a^{-1} \odot a) \odot b & = & (a^{-1} \odot a) \odot c & \text{(property (1) of a group)} \\
e \odot b & = & e \odot c & \text{(property of } a^{-1}\text{)} \\
b & = & c & \text{(property of } e\text{).}
\end{array}$$

The proof is now complete. □

C.10 **Analysis of Proof.** The forward-backward method is used to begin the proof because there are no key words in either the hypothesis or the conclusion. Starting with the forward process, because $a^{-1} \in G$, you can specialize the for-all statement in property (3) of a group to state that

A1: There is an $(a^{-1})^{-1} \in G$ such that $a^{-1} \odot (a^{-1})^{-1} = (a^{-1})^{-1} \odot a^{-1} = e$.

Now, if you can show that $a \in G$ also satisfies the property of $(a^{-1})^{-1}$ in $A1$ then, because the inverse element is unique (see Exercise C.7), by the forward uniqueness method (see Section 11.1), it must be that $(a^{-1})^{-1} = a$. Thus, you must show that

B1: $a^{-1} \odot a = a \odot a^{-1} = e$.

However, $B1$ is true by property (3) in the definition of a group, and so the proof is complete.

Proof. Because $a^{-1} \in G$, by property (3) of a group, there is an element $(a^{-1})^{-1} \in G$ such that $a^{-1} \odot (a^{-1})^{-1} = (a^{-1})^{-1} \odot a^{-1} = e$. Now $a \in G$ also satisfies $a^{-1} \odot a = a \odot a^{-1} = e$. Thus, because the inverse element is unique (see Exercise C.7), it must be that $(a^{-1})^{-1} = a$, completing the proof. $\square$

Appendix D Web Solutions to Exercises

D.5 **Analysis of Proof.** Not recognizing any key words in the hypothesis or conclusion, the forward-backward method is used to begin the proof. A key question associated with the conclusion is, "How can I show that a number (namely, 0) is a lower bound for a set (namely, T)?" Using the definition of a lower bound, you must show that

B1: For all elements $x \in T$, $x \geq 0$.

Recognizing the key words "for all" in the backward statement $B1$, you should now use the choose to choose

A1: An element $x \in T$,

for which it must be shown that

B2: $x \geq 0$.

Working forward from $A1$ using the defining property of T, you know that

A2: $x > 0$ and $x^2 > 2$.

The fact that $x > 0$ in $A2$ ensures that $B2$ is true, thus completing the proof.

Proof. To see that 0 is a lower bound for T, let $x \in T$ (the word "let" here indicates that the choose method is used). But then, by the defining property of T, $x > 0$. This shows that 0 is a lower bound for T, thus completing the proof. $\square$

D.13 **Analysis of Proof.** The words, "Suppose, to the contrary, ..." indicate that the author is using the contradiction method. Accordingly, the author assumes the hypothesis

A: For every real number $\epsilon > 0$, $|x - y| < \epsilon$.

and also

A1 (NOT B): $x \neq y$.

The author then specializes A to the specific value $\epsilon = |x - y|$. To apply specialization, it is necessary to show that this value of ϵ satisfies the certain property in A of being > 0. This, however, is true because $x \neq y$ from $A1$ and so $\epsilon = |x - y| > 0$. The result of specializing $A2$ to $\epsilon = |x - y|$ is that

A2: $|x - y| < |x - y|$.

Now $A2$ is a contradiction because a number (namely, $|x - y|$) cannot be strictly less than itself, thus completing the proof.