Proof-writing style

This document contains some advice on how to write proofs from a stylistic point of view. That is, it doesn't address the actual logical or mathematical content of a proof, only the way in which that content is communicated.

– (S1). The most basic fact about the style of proofs is that a proof is a written text as much as an essay is. It should consist *entirely* of complete sentences in coherent written English[1]. Of course we might use plenty of mathematical symbols, but those symbols should fit properly into honest sentences.

  This doesn't have to be an arduous task. For instance, suppose you want to prove that if $x > 3$, then $\int_2^x 2y - 5\,dy > 0$. Here's a bad proof:

  Proof.

$$\begin{aligned} \int_2^x 2y - 5\,dy &= x^2 - 5x + 6 \\ &= (x-2)(x-3) \end{aligned}$$

  If $x > 3$, then $x - 3 > 0$ and $x - 2 > 0$. Therefore $(x-3)(x-2) > 0$, and $(x-3)(x-2) = \int_2^x 2y - 5\,dy$. □

  Why is this bad? There's a block of equations floating at the top, outside of any sentence. A better version:

  Proof. Notice that

$$\begin{aligned} \int_2^x 2y - 5\,dy &= x^2 - 5x + 6 \\ &= (x-2)(x-3). \end{aligned}$$

  If $x > 3$, then $x - 3 > 0$ and $x - 2 > 0$. Therefore $(x-3)(x-2) > 0$, and $(x-3)(x-2) = \int_2^x 2y - 5\,dy$ by the calculation above. □

[1] In this class, anyway!—though it's an interesting exercise to see how much of a mathematical text written in a foreign language one can puzzle out.

It wasn't very hard to fix the bad proof: we just added a few words to introduce the block of equations, and another few at the end when we wanted to refer back to it. (Admittedly, the "bad proof" wasn't very hard to understand. However, as proofs get more complicated, random blocks of equations get more confusing for the reader without some explicit indication of how they relate to everything else.)

– (S2). Try to avoid beginning sentences with mathematical symbols. We're used to the signposts of capital letters indicating that a new sentence is beginning, and since you wouldn't (I hope!) capitalize a variable or symbol at the start of a sentence, the lack of capitalization can throw off a reader. (See (S5) for a useful way to avoid this.)

  Also, if possible avoid having unrelated mathematical expressions get too close together (e.g. separated only by a period or a comma). This is liable to happen if you start sentences with symbols: "We have shown that $x \in S$. $x \in T$, so $x \in S \cap T$." It happens within sentences too: "Since $x \in T$, $x \in S \cap T$" or "For all $x \in S$, $x > 3$". This is less of an issue since phrases like "Since P, Q" or "For all x, P" are just so convenient and hard to avoid, but you still might want to be aware of it.

– (S3). Don't write an expression half in text and half in symbols, like "Every member of S is > 3". Either say "Every member of S is greater than 3" or "Every $x \in S$ satisfies $x > 3$".

– (S4). Don't overuse symbols. This is always tempting when you've just learned a bunch of new symbols and want to make your proofs look fancy, but a long string of symbols can be hard to read. Personally, I would rather read[2] "Every prime number other than 2 is odd" than "For all primes p, if $p \neq 2$ then $2 \not| p$". Of course, this shouldn't come at the expense of precision or concision: if you can't put a mathematical statement into words without being vague, leave it in symbols, and no one wants to read "The sum of the cubes of the positive integers up to a certain integer equals the square of the sum of the positive integers up to that integer" instead of

$$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2.$$

  One special point to be made here is that the logical symbols $\vee, \wedge, \neg, \exists, \forall, \rightarrow, \leftrightarrow, \equiv, \therefore, \because$ are usually *not* used in formal proofs. The main exception is in logic, when one has to deal with logical formulas in their own right. For example, it's fine to say "The distributive law shows that if P, Q, and R are Boolean variables, then

$$(P \vee Q) \wedge R \equiv (P \wedge R) \vee (Q \wedge R)."$$

  However, you shouldn't use these symbols as part of the ordinary logical flow of a proof. Instead of writing

$$\because p > 2 \vee p \text{ is prime}$$
$$\therefore p \text{ is odd},$$

  write "because $p > 2$ and p is prime, p is odd".

[2]But non-native English speakers may disagree with me here. It's easier to read math in a foreign language when there are fewer words.

- (S5). Occasionally remind the reader what an object is. For instance, imagine that you have invented an object called a "multifuncoid", and that M is the set of multifuncoids. You could say "For every $f \in M$, $f \otimes f \smile {}^{\clubsuit}_{f}\coprod^{\dagger}_{0}$", but it might be more helpful to say "For every multifuncoid $f \in M$, $f \otimes f \smile {}^{\clubsuit}_{f}\coprod^{\dagger}_{0}$". This is redundant, since $f \in M$ already says that f is a multifuncoid, but redundancy can be a good thing.

  Similarly, consider reminding the reader where you are in a proof, or giving an outline of the proof at the beginning (particularly for longer proofs). For instance, "The theorem will follow if we prove P, Q, and R. First we prove P. [proof of P]. Now we prove Q. [proof of Q]. etc.".

- (S6). Introduce a variable before using it, e.g. by binding it to a quantifier or giving it a particular value. For instance, here's a bad proof that $A \cap B \subseteq A \cup B$.

  Proof. Since $x \in A$ and $x \in B$, we know $x \in A$ or $x \in B$. That is, $x \in A \cup B$. □

  But what is x? Are we saying this is true for all x, or is x some particular thing here? A reader who understands how to prove this statement can probably guess that you mean x is an arbitrary member of $A \cap B$, which you will show is a member of $A \cup B$, but of course a proof shouldn't be geared to readers who already know how to do the proof. A better version:

  Proof. Suppose x is in $A \cap B$. Then $x \in A$ and $x \in B$, so $x \in A$ or $x \in B$. That is, $x \in A \cup B$. □

- (S7). Don't write things that aren't true. This may seem obvious, but it's all too easy to be sloppy in the process of transferring a sensible idea in your head onto paper and end up saying something that's false or nonsensical. For instance, if asked to compute the derivative of $x^2/(1+x)$, many calculus students will write

  $$\frac{x^2}{1+x} = \frac{(1+x)(2x) - x^2(1)}{(1+x)^2} = \frac{x^2+2x}{(1+x)^2}.$$

  The first equality is of course totally false, and our hypothetical student should have written

  $$\frac{d}{dx}\frac{x^2}{1+x} = \frac{(1+x)(2x) - x^2(1)}{(1+x)^2} = \frac{x^2+2x}{(1+x)^2},$$

  but somehow got into the habit of using = to express any kind of relationship between two objects without thinking about it.

  An example from our class: I've seen a couple of people write things like "$\varnothing$ is a contradiction". It's easy enough to see that they meant "$x \in \varnothing$ is a contradiction" (which it is), but on its face the first statement is nonsensical, because a contradiction is a type of logical statement, whereas $\varnothing$ is a set, not a logical statement. If you're familiar with computer programming, you might imagine this as being careful about type checking: $\varnothing$

is a different type of object from a logical statement, and "contradiction" applies only to the latter type.

(See also item (T2) below.)

- (S8). Watch out for ambiguous pronouns ("it", "they", "that", "this", etc.). For example, consider the following proof that if $x, y \in \mathbb{R}$, x is an integer, and y is not an integer, then $x + y$ is not an integer.

  Proof. Suppose that $x \in \mathbb{R}$ and $y \in \mathbb{R} \setminus \mathbb{Z}$. If $x + y \in \mathbb{Z}$, then $(x + y) - x = y \in \mathbb{Z}$. This is a contradiction, so it is not in $\mathbb{Z}$. □

  But what is "it" in the last sentence? Is it $x + y$ or $(x + y) - x$? From context we can guess "it" probably means $x + y$, but this sort of ambiguity only gets deadlier as proofs get more complicated. (This isn't to say that you can't use pronouns, just that they should be unambiguous. "If x is the square of a real number, then it is nonnegative" is fine.

- (S9). Watch out for overly long sentences, or those with complicated structures. It's easy to get carried away with the flow of an argument and write "Since $x \in A$, we know $x^2 \in B$, so $x + x^2 \notin A \setminus C$, which means $\int_x^{x^2} \phi(y^y)\, dx = \pi$, but this is a contradiction, so we conclude that x is irrational.". Reading mathematics is already hard enough on its own, so give readers a break and try to keep sentences reasonably short and simple.

- (T1). Let vs. suppose vs. assume. "Suppose" and "assume" mean more or less the same thing: that we will be assuming a particular fact. Thus one can write "Suppose $x = 3$" or "Suppose $A \cap B \subseteq B \cap C$" or "Assume that there are finitely many prime numbers".

  "Let" has a more restrictive meaning: it can *only* be used with a variable, to say what value that variable will have or what set it comes from. Thus we can say "Let $x = 3$" or "Let x be a real number", but we don't say "Let $A \cap B \subseteq B \cap C$ or "Let there be only finitely many prime numbers".

- (T2). Keep logical symbols and set theory symbols distinct. This is perhaps a special case of item (S7) above, but is a common enough mistake that it seems worth discussing separately. The symbols $\vee, \wedge, \neg, \rightarrow, \leftrightarrow$ apply to logical statements, not sets. The symbols $\cup, \cap, \setminus, {}^c, \subseteq$ apply to sets, not logical statements. These two sets of symbols are nicely related, in the sense that (for example)

  $$x \in A \cup B \text{ is equivalent to } x \in A \vee x \in B,$$

  but they don't apply to the same types of objects.