Temporal Latent Space Network Model with VAR 1 Evolution of Latent Positions: Introduction

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1 Motivation and Introduction

In the latent space network model, the underlying network structure is represented by the positions of nodes in a continuous (Euclidean) latent space. This class of model allows for the basic network properties like reciprocity and transitivity of the nodes, with possible extension to clusterability. We aim to extend the idea of reciprocity and transitivity in a static network to the temporal network settings. The nodes with ties at previous time points are more likely to have ties in the future, indicating that they will lie close to each other in the future latent space. Similary, if the nodes i and j, and the nodes i and k have ties at time t, the nodes i and k are more likely to have a tie at $t + 1$. Thus, it is important to extend the static latent space model to account for these possibilities of forming ties in the longitudinal network data. Furthermore, a network can evolve in time by the expansion of the latent positions making the network sparser, by the shrinkage of the latent positions which makes the network denser, or by no change in the latent positions which implies no substantive change in the network structure, after accounting for a common feature of the networks that do not change over time. Most of the existing statistical methods for network analysis focus on a single network observed at a time point [[TODO:CITE]]. These methods, that treat related networks to be independent over time, are not well-suited for networks observed over multiple time points. By modeling the temporal networks as different instances of static networks, we lose important information about network evolution. Instead, modeling them as a continous process that accounts for the time dependencies helps us understand the evolution of the underlying network structure. In this work, we propose a state space modeling approach for evolution of temporal networks using a class of latent variable models. More specifically we focus on VAR (1) representation of the evolution rather than random walk process to model a stable and stationary evolution process that is commonly seen in real world social networks.

New paragraph?

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What is VAR?

Maybe mention independence? I was expecting to see it in this paragraph, but it never came up.

Recently, there has been some work on extending existing network methods to account for temporal networks, for example by Robins and Pattison (2001), Hanneke and Xing (2007),Hanneke et al. (2010), Westveld and Hoff (2011), Xing et al. (2010), Sarkar and Moore (2005) and Sewell and Chen (accepted). Robins and Pattison (2001), Hanneke and Xing (2007) and Hanneke et al. (2010) have studied the networks observed over discrete time points in the ERGM settings, also known as ***temporal ERGMs or TERGMs***. TERGMs make standard Markov assumption on the evolution of a network graph such that Y_t is independent of $Y_1, \ldots, Y_{t-2}$ given Y_{t-1} (Hanneke and Xing, 2007), with an additional assumption that $P(Y_t|Y_{t-1})$ has an ERGM representation [[TODO:CITE ERGM PAPER HERE]]. Example of network statistics can include statistics representing stability, density, overall reciprocity, etc observed in the networks at time t and $t - 1$. Similar to the static ERGM, assumptions on the dependence structure of ties for a network at time t conditional on the network at time $t - 1$ will influence the potential network statistics that can be included in the model. As the assumption on the tie dependence deviates from independence, the model gets more and more complicated. Snijders (1996) have also developed stochastic actor oriented models using a continuous time Markov processes, the class of models that is very similar to ERGMs. Westveld and Hoff (2011) extended the static model for directed networks with sender and receiver random effects and fixed covariate effects, introduced by Gill and Swartz (2001) and later implemented by Hoff and Ward (2004), to account for temporal dependencies in ***mixed effects temporal model***. They assume autoregressive dependence structure on the sender-receiver effects and the overall residual, and hence account for additional correlation in random error (random effects) introduced by temporal dependencies.

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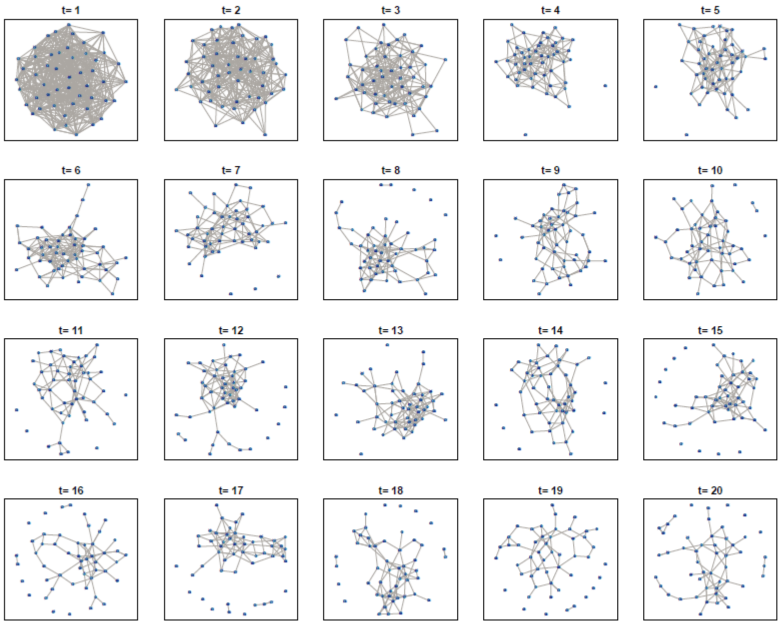
What is ERGM? Maybe you don't need to define it here because it's obvious to the audience.

Why do you bold and italicize these terms? Maybe you could just bold them.

Maybe you can split this into two sentences?



$$
\begin{aligned}
Y_{ijt} &\sim \mathrm{Bernoulli}(p_{ijt}) \\
p_{ijt} &= \beta_0 - ||Z_{it} - Z_{jt}|| \\
Z_{it} &\sim \mathrm{Normal}(Z_{i(t-1)}, \Sigma) \\
\beta_0 &- 0.1 \\
\Sigma &= \begin{pmatrix} 0.75 & 0 \\ 0 & 0.75 \end{pmatrix} \\
T &= 20 \\
n_t &= 50, \forall T \\
dd &= 2
\end{aligned}
$$

Figure 1: Networks simulated under random walk plus noise evolution of the latent positions

Xing et al. (2010) extended ***mixed membership stochastic block model (MMSBM)*** (Airoldi et al., 2006) to account for the temporal nature of networks, and called it the ***dynamic mixed membership stochastic block model (dMMSBM)***. Xing et al. (2010) developed a Bayesian state-space approach for modeling the evolution of the underlying roles of entities in a network, such that a network evolves in time through the random walk dependence structure on the hyper-parameters of the prior distribution of the membership vectors and the block probabilities. Sarkar and Moore (2005) introduced a predictive latent space model for temporal networks. We also note that similar work is being developed independently by Sewell and Chen (accepted). Both Sarkar and Moore (2005) and Sewell and Chen (accepted) focus on random walk plus noise evolution.

Maybe you could say this earlier? You could have a paragraph that summarizes what you will do, and then you go on to talk about what others have done.

In this paper, we focus on ***latent space network models*** which assume that the nodes in a network have underlying positions in a low dimensional Euclidean space. Hoff et al. (2002) introducted latent space model for social networks, and Hoff and Ward (2004); Handcock et al. (2007); Raftery et al. (2012) have explored methodological and computational aspects of the static latent space models. We assume that the network dynamics in time are direct functions of the latent nodal positions which show stationary vector autoregressive evolution of order 1. Existing temporal methods for latent space model focus on random walk evolution of latent positions. While these models are a good starting point they do not necessarily account for many different types of dynamics that we see in real world networks. Since the variance of the latent positions are increasing with time in random walk models thus implying that the network gets sparser with time, this assumption is not realistic in social networks, for example advice seeking network of teachers, where while the nodes move around in the space they do not necessarily keep moving away from each other. [[TODO:CITE]] In this paper, we introduce an alternative way to specify the evolution of the latent positions as a stable and stationary process. We also present an MCMC algorithm to draw samples from the posterior distribution of the parameters. Latent space positions are identifiable in this model only upto the class of distance preserving transformations. This unidentifiability in the latent space positions also contributes to unidentifiability of the VAR parameter. The novel contribution of this work is in specifying the identifiable component of the VAR parameter and its significance in understanding the evolution of the networks with time.

Separate paragraph? This is important because it's your contribution and why it's necesary.

1.1 Notations and network terminologies

Let Y denote a random variable representing a network graph. We use upper-case Y to denote a random variable, and lower-case y to denote its realization. For a static network, ***an observed network*** is a graph with n vertices. These vertices are called ***the nodes*** of the network, and the corresponding edges connecting the vertices are ***the ties***. An observed network is usually represented as a $n \times n$ sociomatrix y with entries y_{ij}, where y_{ij} measures the strength of a relationship from node i to node j, and can be either discrete or continuous. We call y_{ij} a tie from i to j. A tie y_{ij} and its reciprocal tie y_{ji} collectively form ***a dyad*** between i and j. For an undirected network, y_{ij} is equal to y_{ji}.

We will use Z to denote a $n \times d$ matrix of the latent positions, such that its i^{th} row Z_i is a vector representing the position of a node i in a d dimensional latent space. Let p_{ij} denote the probability of forming a tie between nodes i and j and $d(Z_i, Z_j)$ denote the distance (for example, Euclidean) between the latent positions Z_i and Z_j.

In the case of a temporal network, we define y_t as a sociomatrix of the observed network at time t with entries y_{ijt} measuring a relationship from node i to j at that time point. Further, we will use $y_{1:T} := [y_1 y_2 \ldots y_T]$ to denote the block matrix of socio-matrices observed upto T time point, and $Z_{1:T} := [Z_1 Z_2 \ldots Z_T]$ to denote the block matrix of latent positions upto time T. y_t is a $n \times n$ matrix of ties, with NA along the diagonal. Z_t is a $n \times d$ matrix of d dimensional latent space, with z_{it} denoting i^{th} row of Z_t. Finally, β_0 is an overall intercept of the model.

For simplicity, we will consider a discrete and undirected network Y such that

$$\begin{cases} y_{ij} & = 1 \text{ if there is a tie between i and j} \\ & = 0 \text{ if there is no tie.} \end{cases}$$

Bracket should go right before cases: http://latex.wikia.com/wiki/Cases_(LaTeX_environment)

However, these models can be easily extended to ordinal and continuous valued ties based on the techniques used for generalized linear models.

2 Model

The ***latent space network model (LSM)*** introduced by Hoff et al. (2002) is characterized by the positions of the nodes in a low-dimensional latent space. Hoff et al. (2002) describe the latent space as a social space containing the unobserved characteristics of the network, where nodes with similar latent characteristics will have nearby latent positions. Further, conditional on the latent positions the ties in a network are assumed to form independently, and the probability of a tie between nodes i and j is inversely related to the interdistance between their latent positions. The implications of the assumption are: i. if two nodes share a tie, they lie close to each other in the latent space and, ii. if two ties in a network share a common node, then the two remaining nodes will lie close to each other in the latent space hence increasing the probability of a tie between them. Thus, LSM accounts for the two basic network properties, ***transitivity and reciprocity***, which is described in more detail in Hoff et al. (2002).

New paragraph?

We can represent the model in notation in Equation 1:

$$\begin{aligned} y_{ij} &\sim \text{Bernoulli}(p_{ij}) \\ \eta_{ij} &:= \text{logit}\,(p_{ij}) = \beta_0 - ||Z_i - Z_j|| \\ Z_i &\sim MVN(0, \Sigma). \end{aligned} \tag{1}$$

Further, the likelihood of the observed network y conditional on the latent positions Z and the intercept β_0 is then given by Equation 2:

$$P(Y = y|Z, \beta_0) = \prod_{i \neq j} exp[\eta_{ij} y_{ij} - \log{(1 + exp(\eta_{ij}))}]. \tag{2}$$

The intercept β_0 in the model can be seen as an overall fixed network effect, whereas the latent positions Z_is are the random effects. Further, it is evident from the model presented above that if the two nodes i and j have the same latent position, then the log-odds of forming a tie between i and j is β_0. LSM is arguably a useful and appealing method for network analysis because it implicitly models different network features while making few assumptions about the dependence structure of the ties.

In this paper, we combine ideas from the latent space model of Hoff et al. (2002) and the state-space modeling approach [[TODO:CITE]] to model the evolution of networks in time through the changes in latent positions. We allow autoregressive dependence of order 1 in the nodal positions as a function of the previous latent positions.

Then, the LSM for static model can be extended to account for temporal features in Equation 3 :

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$$
\begin{aligned}
y_{ijt} &\sim Bernoulli(p_{ijt}) \text{ for } i \neq j \\
logit(p_{ijt}) &= \beta_0 - ||Z_{it} - Z_{jt}|| \\
Z_{i,1} &\sim MVN(0, \Sigma_0), \text{ for } i = 1, \dots, n \\
Z_{i,t} &= \Phi Z_{i,t-1} + \epsilon_t \text{ for } t = 2, \dots, T \\
\epsilon_t &\sim MVN(0, \Sigma).
\end{aligned} \tag{3}
$$

Further, assuming stationary VAR model we can compute the covariance matrix of each $Z_{i,t}$ as

$$vec(\Sigma_0) := vec(var(Z_{i,t})) = var(Z_{i,1}) = (I - \Phi * \Phi)^{-1} vec(\Sigma).$$

[[TODO:CITE]]

Here, $A * B$ is a Kronecker Delta product of two matrices A and B, I is an identity matrix of dimension $d^2 \times d^2$ and $vec(\Sigma)$ is a vector formed by stacking columns of Σ together.

Now lets look closely at the stationarity condition of centered VAR of order p, which can be defined as in Equation 4:

$$
\begin{aligned}
Z_{it} &= \Phi_1 Z_{i(t-1)} + \Phi_2 Z_{i(t-2)} + \dots + \Phi_p Z_{(i(t-p)} + \epsilon_{it} \\
Z_{it} - \sum_{j=1}^{p} \Phi_j Z_{i(t-j)} &= \epsilon_{it} \\
Z_{it} - \sum_{j=1}^{p} \Phi_j B^j Z_{it} &= \epsilon_{it} \\
(I - \sum_{j=1}^{p} \Phi_j B^j) Z_{it} &= \epsilon_{it}
\end{aligned} \tag{4}
$$

Here, B is the backshift operator and I is $d \times d$ identity matrix. Then we have the condition that Z_{it} is a stationary VAR(p) process if the roots of the $det\{I - \sum_{j=1}^{p} \Phi_j B^j\}$ are all outside the unit circle or equivalently all are greater than 1 in absolute value. For $VAR(1)$ process this condition is satisfied if the eigen values of Φ are less than 1 in absolute value.

One word

3 Estimation

We will use Metropolis Hastings within Gibbs algorithm to draw samples from the posterior distribution of the parameters, namely, β_0, $Z_{1:T}$, Φ and Σ. First, lets look at the full-likelihood of Y, Z, β_0 and Φ under the model illustrated in Equation 3:

$$
\begin{aligned}
P(Y, Z, \beta_0, \Phi) &= P(Y|Z_1, \dots, Z_T, \beta_0, \Phi) P(Z_1, \dots, Z_T, \beta_0, \Phi) \\
&= \prod_{t=1}^{T} P(Y_t|Z_t, \beta_0) \times P(Z_1|\mu_z, \Sigma, \Phi) \times \prod_{t=2}^{T} P(Z_t|\Phi, Z_{t-1}, \Sigma) \times P(\beta_0) \times P(\Phi) \times P(\Sigma)
\end{aligned} \tag{5}
$$

The first term in the product in Equation 5 can be easily obtained from Equation 2 for each time point t since the networks are independent over time conditional on the latent positions and the intercept. However, we need to account for the changing number of the nodes over time while computing the likelihood of the latent positions. We will assume that nodes are missing at random. Further, once a node exits a network there is a very less chance that it will re-enter. Thus, if a node enters the networks after $t = 1$ we will use the time point as its initial time and assume that its latent position has the same prior as Z_{i1}. Let $\{N_{t-1}\}$ denote set of nodes at time t that were also present at $t - 1$.

The likelihood in Equation 5 can be re-written as:

$$
\begin{aligned}
P(Y, Z, \beta_0, \Phi) = & \prod_{t=1}^{T} \prod_{i \neq j} P(Y_{i,j,t} | z_{i,t}, z_{j,t}, \beta_0) \times \prod_{i=1}^{n_1} P(z_{i,1} | \mu_z, \Sigma_0(\Phi)) \\
& \times \prod_{t=2}^{T} \prod_{i=1}^{n_t} [P(z_{i,t} | \Phi, z_{i,t-1}, \Sigma) \mathsf{I}(i \in \{N_{t-1}\}) + P(z_{i,t} | \mu_z, \Sigma_0(\Phi)) \mathsf{I}(i \in \{N_{t-1}\})] \\
& \times P(\beta_0) \times P(\Phi) \times P(\Sigma)
\end{aligned}
\tag{6}
$$

[[TODO:Include algorithm to illustrate MCMC step]]

3.0.1 Orientation of the latent positions from $t = 1$ up to $t = T$ within a MCMC step

In Equation 6, note that the first term is invariant to the distance preserving transformations in Z_t. The loglikelihood of the network is related to the latent positions only through the interdistance of the nodes in the latent space, hence the first term in the above equation is invariant to the isometric transformations in Z_t's. However, the multivariate normal density function is not invariant to the isometric transformation (unless the variance covariance matrix is diagonal representing a spherical distribution) (Tong, 2012). In this section, we will explore whether rotation of the latent positions at each time point within a single Monte Carlo run will be an issue.

[[TODO:CITE/ Acknowledge Cosma for this proof]] Let Z_t and X_t be two isometric latent configurations. We may fix, within a single Monte Carlo run, the previous latent configuration Z_{t-1}, the evolution operator Φ, and the noise variance of the latent evolution Σ.

Claim: If Σ^{-1} is strictly positive definite, then for Lebesgue-almost-all X_t isometric to Z_t, $Pr(X_t|Z_{t-1}, \Phi, \Sigma) \neq Pr(Z_t|Z_{t-1}, \Phi, \Sigma)$.

Proof: The two probabilities are equal iff their logarithms are equal. After canceling constant terms which are the same between the two probabilities, the logs are equal iff

$$(X_t - \Phi Z_{t-1})^T \Sigma^{-1} (X_t - \Phi Z_{t-1}) = (Z_t - \Phi Z_{t-1})^T \Sigma^{-1} (Z_t - \Phi Z_{t-1}).$$

Since (by assumption) Σ^{-1} is positive-definite, the quadratic forms appearing on either side of the equation are both ≥ 0. If only one is zero, there can't be equality, so either they're both zero or they're both positive. If they are both zero, then (again by positive-definiteness of the matrix) $X_t - \Phi Z_{t-1} = 0 = Z_t - \Phi Z_{t-1}$, implying $X_t = Z_t$, which contradicts the assumption that they are distinct.

Then, we are left with the case where both the quadratic forms equal the same positive number, lets call it c. The set of points in the latent space where the quadratic form is equal to c is an ellipsoid, centered at ΦZ_{t-1}, where the directions of the axes come from the eigenvectors of Σ and their widths from the eigenvalues. An important point here is the center, and the fact that, because the eigenvalues are all positive, the surface extends through all dimensions of the latent space.

Our concern is that X_t might be an -arbitrary- isometry of Z_t. But since X_t and Z_t must both lie on the ellipsoid, which is centered at ΦZ_{t-1}, they cannot be related through arbitrary translations. Thus we only need to concern ourselves with the case $X_t = RZ_t$, for an arbitrary rotation-reflection isometry R. But the set of such X_t consists of a sphere centered at the origin, which cannot coincide with, or even be a subset of, an ellipsoid centered elsewhere. There may be a set of intersection between the sphere and the ellipsoid, but (being the intersection of two low-dimensional surfaces) it will have Lebesgue measure 0. □

Thus, from the above proof we observe that while rotation of the latent positions at the first time point for each Monte Carlo draw is a possibility, the orientation of the positions for the subsequent time points at that draw will only depend on the orientation of the first one. We attempt to fix for that rotation by using Procustean transformation of the latent positions at the first time point to a fixed target for each MCMC draws.

Next we will have results on how this tranformation affect the estimation of Φ parameter.

3.1 Identifiable Component of Φ in MCMC Estimation

Let us begin by rewriting the evolution model (state equation) that relates latent space positions at time t and time $t-1$:

$$Z_t = Z_{t-1}\Phi^T + \epsilon_t \tag{7}$$

We will frequently refer to 7 as an equation relating true underlying relationship between latent positions over time. Our goal in the estimation is to recover and understand this relationship. However, rotational invariance of the model while sampling the latent positions in this setup leads to unidentifiability in the Φ parameter. In this section, we introduce a way to estimate a new matrix, lets call it Φ^*, that is a similar matrix to Φ.

Lets consider the K^{th} draw of the latent positions for all T times points in MCMC estimation. We will denote it by $\hat{Z_{1:T}}^K$. The latent positions at $t = 1$ can only be estimated upto an isometry as the latent distance is invariant to such transformation, and the orientation of $\hat{Z}_1$ also controls the orientation of $\hat{Z}_t$s for $t > 1$. $\hat{Z}_{1:T}^K$ is some isometric transformation of the true positions Z_t that satisfies the relationship in 7, upto MCMC error. Let L_t^K denote the transformation operator at each time t during the estimation. Also, note that $L_t^K = L_{t-1}^K \forall t$. However, internodal distance is preserved during these transformations (upto MCMC error). If we use $D(.)$ to denote the Euclidean distance operator for a matrix . then we can write:

$$D(L_t^K(Z_t)) = D(Z_t)\forall t.$$

Because of this rotational effect during the estimation we cannot estimate the true Φ. Further, since $Z_{1:T}$ are unknown (latent) parameters, $L_{1:T}^K$ are also unknown. We attempt to fix this problem by using procrustes transformation of the latent space positions, $\hat{Z}_1^K$, at each step of the Monte Carlo draw.

Let Z_{00} denote a fixed set of positions for n nodes in a d dimensional Euclidean space. We will call Z_{00} our ***target positions***. Then, at each step of MCMC we do procrustes transformation on $\hat{Z}_1^K$ such that they are as close to Z_{00} as possible while preserving the interdistances between the nodes. Lets denote this isometric transformation in the K^{th} MCMC draw by P_1^K. Note that $P_1(.)^K$ is a function of A and $\hat{Z}_1^K$, as the transformation matrix for each time point depends on target A and $\hat{Z}_1^K$. (Refer to section on Procrustean transformation.)

In the next two Lemmas, we first show that doing procrustes transformation on $\hat{Z}_1^K$ produces the same set of positions as does the procrustes transformation on Z_1 directly. This fact justifies the use of procrustes transformation within our estimation method. Next, we show how the VAR parameter in the transformed positions are related to the true Φ.

Theorem 3.1 *Let Z_{00} be an arbitrary (centered) positions. For each $\hat{Z}_1^K$, let P_1^K denote a isometric transformation such that:*

$$P_1^K := argmin_T tr(Z_{00} - \hat{Z}_1^K T)(Z_{00} - \hat{Z}_1^K T)^T.$$

For each Z_1, let T_1^* denote a isometric transformation such that:

$$T_1^* = argmin_T tr(Z_{00} - Z_1 T)(Z_{00} - Z_1 T)^T.$$

Then, $T_1^* = L_1^K P_1^K$.

Proof Let $SVD(Z_{00}^T Z_t) = U\Lambda V^T$.

Then, $T_1^* := \text{argmin}_T tr(Z_{00} - Z_1 T)(Z_{00} - Z_1 T)^T = VU^T$.

Also,

$$\begin{aligned} Z_{00}^T Z_1 L_1^K &= SVD(Z_{00}^T Z_1) L_1^K \\ &= U\Lambda V^T L_1^K \\ &= U\Lambda V_1^{*KT} \\ &\quad (V_t^{*KT} \text{ is an orthogonal matrix}) \\ &= SVD(Z_{00}^T Z_1 L_1^K). \end{aligned}$$

Then, $P_1^K := \text{argmin}_T tr(Z_{00} - TZ_1L_1^K) = V_1^{*K}U^T = L_1^{TK}VU^T$.

Finally note that: $L_1^K P_1^K = L_1^K L_1^{KT} VU^T = VU^T = T_1^*$. $\blacksquare$

Theorem 3.2 *Let $SVD(\Phi) = U\Lambda V^T$ and $SVD(\Phi^{*K})$ = $U^{*K}\Lambda^{*K}V^{*TK}$ where, U, V, U^{*K} and V^{*K} are orthogonal matrices and Λ and Λ^{*K} are diagonal matrices of singular values.*

If $\hat{Z}_t^K = \hat{Z}_{t-1}^K(\Phi^{*K})^T + \epsilon_t^{*K}$, then Φ and Φ^{*K} are similar matrices. Further, $\Lambda = \Lambda^{*K}$.

Proof Now the new dependence equation between transformed latent space positions becomes (which is constant upto some MCMC error for each draw in MCMC):

$$\begin{aligned}
P_t(W_t^K) &= P_{t-1}(W_{t-1}^K)\Phi^{*KT} + \epsilon_t^{*K} \\
P_t(L_t(Z_t)) &= P_{t-1}(L_{t-1}(Z_{t-1}))\Phi^{*T} + \epsilon_t^{*K} \\
Z_t(P_t^K L_t^K)^T &= Z_{t-1}(P_{t-1^K} L_{t-1}^K)^T \Phi^{*KT} + \epsilon_t^{*K} \\
Z_t &= Z_{t-1}(P_{t-1}^K L_{t-1}^K)^T \Phi^{*KT}(P_t^K L_t^K) + \epsilon_t^{*K} P_t^K L_t^K
\end{aligned}$$

The above equations give us tools to relate Φ^* and Φ such that:

$$\Phi = (P_{t-1}^K L_{t-1}^K)^T \Phi^{*K}(P_t^K L_t^K). \tag{8}$$

First note that, $P_{t-1}L_{t-1}$ and P_tL_t are both rotation matrices. Also,

$$\begin{aligned}
(P_{t-2}^K L_{t-2}^K)^T \Phi^{*K} P_{t-1}^K L_{t-1}^K &= (P_{t-1}^K L_{t-1}^K)^T \Phi^{*K}(P_t^K L_t^K) \\
\Rightarrow \Phi^{*K} &= (P_{t-2}^K L_{t-2}^K)(P_{t-1}^K L_{t-1}^K)^T \Phi^{*K}(P_t^K L_t^K)(P_{t-1}^K L_{t-1}^K)^T
\end{aligned}$$

Thus we have that,

$$\begin{aligned}
(P_{t-2}^K L_{t-2}^K)(P_{t-1}^K L_{t-1}^K)^T &= I \\
\Rightarrow P_{t-2}^K L_{t-2}^K &= ((P_{t-1}^K L_{t-1}^K)^T)^{-1} \\
&= ((P_{t-1}^K L_{t-1}^K)^T)^T \\
&= (P_{t-1}^K L_{t-1}^K)
\end{aligned}$$

We can similarly show that $P_t^K L_t^K = P_{t-1}^K L_{t-1}^K = C^K$ for all $t = 1, \ldots, T$, where C^K is an orthogonal matrix. Putting this all together we can rewrite 8 as:

$$\Phi = C^{KT}\Phi^{*K}C^K = (C^K)^{-1}\Phi^{*K}C^K.$$

Thus, we showed that Φ and Φ^{*K} are similar matrices.

Next, denote $SVD(\Phi) = U\Lambda V^T$ where U and V are orthogonal matrices. Also, denote $SVD(\Phi^*)$ by $U^*\Lambda^*V^{*T}$. Since, $SVD(\Phi)$ is unique given some regulatory conditions on Φ we can show that $\Lambda = \Lambda^*$. Further,

$$(P_{t-1}L_{t-1})^T U^* = U$$

and

$$(P_tL_t)V^{*T} = V^T$$

Further, the variance covariance matrix of ϵ_t^* is also transformed such that $COV(\epsilon_{it}^* P_tL_t) = (P_tL_t)\Sigma^*(P_tL_t)^T = COV(\epsilon_{it})$. In some ways, this equation gives us a way to define a distribution of ϵ_t.

3.2 Prior specification

The priors in the model are specified as below:

$$\beta_0 \sim \text{Normal}(\mu_0, \sigma_0^2)$$
$$\Sigma_{ii} \sim \text{InverseGamma}(A, B)$$
$$[\Phi_{ij}] \sim \text{Normal}(\mu, \tau)\mathbf{I}\{|\lambda_{ii}| < 1\} \;\; \forall ii$$

where, $[\Phi_{ij}]$ is the i, j^{th} entry og Φ and λ_{ii} is the ii^{th} eigen value of Φ.

μ_z,Σ_0,μ_0,σ_0^2 are hyperparameters of the model specifying mean and variance of the prior distribution of latent positions at time $t = 1$ and intercept respectively. A and B are shape and rate parameters of the Inverse-Gamma prior distribution on the variance of the error term in the latent space positions.

4 Prediction and Model Comparision

One of the goals of this work is to develop a systematic approach to predict future ties at T_1 given the networks upto time T. We can use the posterior MCMC draws of $Z_{1:T}$ to estimate $\hat{Z}_{1:T}$. Observe that, for VAR(1) evolution model:

$$\begin{aligned} P(Z_{T+1}|Y_{1:T} &= \int P(Z_{T+1}|Z_{1:T}, \beta_0, \Phi, \Sigma)P(Z_{1:T}|Y_{1:T}, \beta_0, \Phi, \Sigma)dZ_{1:T}d\beta_0 d\Phi d\Sigma \\ &\approx \frac{1}{L}\sum_{l=1}^{L}\prod_{i=1}^{n_{T+1}}[N(Z_{i(T+1)}|\Phi Z_{iT}^l, \Sigma^l)\mathsf{I}(i \in \{N_T\}) + N(Z_{i(T+1)}|\mu_z, \Sigma_0^l(\Phi^l))\mathsf{I}(i \notin \{N_T\})] \end{aligned} \tag{9}$$

Then,

$$\hat{Z}_{T+1} = \mathsf{E}(Z_{T+1}) = \begin{cases} \frac{1}{L}\sum_{l=1}^{L}(\Phi^l Z_{iT}^l) & \text{if}(i \in \{N_T\}) \\ \mu_z & \text{else} \end{cases}.$$

And finally the predictive probability of tie between nodes i and j at time $T + 1$ is

$\hat{p}_{ij(T+1)} = \hat{\beta}_0 - ||\hat{Z}_{i(T+1)} - \hat{Z}_{j(T+1)}||$.

We use similar method for prediction using estimates of latent space positions from random walk model.

To make latent space model comparable with other longitudinal data we use $Y_{1:T}$ to draw intercept and Σ in the following way:

4.1 Estimation with Missing Nodes

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