36-752: Advance Probability	**Spring 2018**

Lecture 15: March 27

Lecturer: Alessandro Rinaldo *Scribes: Riccardo Fogliato*

Note: *LaTeX template courtesy of UC Berkeley EECS dept.*

Disclaimer: *These notes have not been subjected to the usual scrutiny reserved for formal publications. They may be distributed outside this class only with the permission of the Instructor.*

Example 17: Consider the measure space $(\Omega, \mathcal{F}, P)$, where $\Omega = (0,1)$ and P is the Lebesgue measure. Define the following series of functions:

$$f_1 = 1, \quad f_2 = \mathbb{1}_{(0,1/2]}, \quad f_3 = \mathbb{1}_{[1/2,1)}, \quad f_4 = \mathbb{1}_{[2/3,1)}, \quad \dots$$

Then

$$f_n \xrightarrow{L^p} 0 \;\forall p,$$

but f_n does not convergence to 0 since $\limsup f_n \neq \liminf f_n$.
Example 18: Consider

$$f_n = \begin{cases} 0 \text{ if } 0 < \omega < \frac{1}{n} \\ \frac{1}{\omega} \text{ if } \omega \geq \frac{1}{n} \end{cases}$$

Then

$$\lim_{n\to\infty} f_n(\omega) = \frac{1}{\omega} \quad \forall \omega \in (0,1)$$

but f_n does not converge in L^p even if $f_n \in L^p \;\forall n$.
Example 19: Consider

$$f_n(\omega) \begin{cases} n \text{ if } 0 < \omega < \frac{1}{n} \\ 0 \text{ if } \omega \geq \frac{1}{n} \end{cases}$$

Then $f_n \xrightarrow{as} 0$ and $Y_n \xrightarrow{p} 0$ where $Y_n = f_n(X)$ with X being a uniformly distributed random variable on $[0,1]$. However, $\int_0^1 |f_n| dx = n \;\forall n, p \geq 1$; hence f_n does not convergence in L^p to 0.

Claim 15.1 *L^p convergence implies convergence in probability.*
Almost sure convergence implies convergence in probability.

Proof: First part: prove it using Markov's inequality.
Second part: fix $\epsilon > 0$, $A_{n,\epsilon} = \{|X_n - X| > \epsilon\}$. Then $A_{n,\epsilon}$ is a decreasing sequence so that

$$\lim_{n\to\infty} \mathbb{P}(A_{n,\epsilon}) = \limsup_n \mathbb{P}(A_{n,\epsilon}) \leq \mathbb{P}(\limsup_n A_{n,\epsilon}) = 0$$

by Fatou's lemma. ■

Lemma 15.2 *Let $\{f_n\}$ and f be non negative functions in $L^1(\Omega, \mathcal{F}, P)$. If $f_n \xrightarrow{as} f$, then $\int |f_n - f| dP \to 0$ if and only if $\int f_n dP \to \int f dP$ (L^1 convergence).*

Proof: First part:

$$|\int f_n - \int f|dP \leq \int |f_n - f|dP \to 0.$$

Second part: let $g_n = \min\{f_n, f\}$; then $g_n \leq f$ a.e. and $\int g_n \to \int f$ by dominated convergence theorem. Then

$$\int |f_n - f|dP = \int f_n dP + \int f dP - 2\int g_n dP \to \int f dP + \int f dP - 2\int f dP = 0.$$

■

Corollary 15.3 *Let μ_n and μ be measures on $(\Omega, \mathcal{F}, \nu)$ with $\mu_n(A) = \int_A f_n d\nu_n$ and $\mu(A) = \int_A f\nu\ \forall A \in \mathcal{F}$. If $\mu(\Omega) = \mu_n(\Omega) < \infty$ and $f_n \xrightarrow{as} f$, then*

$$\sup_{A\in\mathcal{F}} |\mu_n(A) - \mu(A)| \leq \int |f_n - f|d\nu \to 0.$$

15.1 Strong law of large numbers

Definition 15.4 *(Tail $\sigma-$field) Let $\tau_n = \sigma(\{X_i, i \geq n\})$. The tail σ-field is defined as $\tau = \cap_n \tau_n$*

Events in τ are not affected by changes in a finite number of terms in the sequence.
Some examples of tail events:

- $\{\lim_n X_n = x\}$;
- $\{\sum_n X_n \text{ converges }\}$;
- $\sup_n X_n$ is measurable with respect to τ.

Theorem 15.5 *(Kolmogorov 0-1 law) Let $\{X_n\}$ be a sequence of independent random variables. If $A \in \tau$, then $\mathbb{P}(A)$ is either 1 or 0.*

Proof: Let $U_n = \sigma(\{X_i : i \leq n\})$ and $U = \cup_n U_n$. Then U is not a σ-field. Let $B \in U, A \in \tau$, then $B \in U \Rightarrow \exists n_0$ s.t. $B \in U_{n_0}$. But since $A \in \tau$, then $A \in \tau_{n_0+1}$. Then

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B),$$

ie U and τ are independent, hence $\sigma(U)$ is independent of τ.
On the other hand, $\tau \subset \sigma(U)$; hence τ is independent of itself. Therefore

$$\mathbb{P}(A \cap A) = \mathbb{P}(A)\mathbb{P}(A)\ \forall A \in \tau.$$

This equality is satisfied if and only if $\mathbb{P}(A)$ is either 0 or 1. ■

Theorem 15.6 *(First Borel Cantelli Lemma) Let $\{A_n\}$ be a sequence of events in $\mathcal{F}$ such that $\sum_{n\geq 1} \mathbb{P}(A_n) < \infty$. Then*

$$\mathbb{P}(\limsup A_n) = 0$$

Proof: Let $B_i = \cup_{n=i}^{\infty} A_n$. Then B_i is a decreasing sequence, and

$$\lim_n \mathbb{P}(B_i) = \mathbb{P}(\lim B_i) = \mathbb{P}(\cap_i B_i) = \mathbb{P}(\limsup A_n).$$

Then

$$\mathbb{P}(B_i) \leq \sum_{n=i}^{\infty} \mathbb{P}(A_n) \to 0 \text{ as } i \to \infty.$$

■

Theorem 15.7 *(Second Borel Cantelli Lemma) Let $\{A_n\}$ be a sequence of independent events in $\mathcal{F}$ and $\sum_{n\geq 1} \mathbb{P}(A_n) = \infty$. Then $\mathbb{P}(\limsup A_n) = 1$.*

Corollary 15.8 *If $X_n \xrightarrow{p} X$, then $\exists \{n_k\}$ st. $X_{n_k} \xrightarrow{as} X$.*

Proof: Wlog consider $n_k > n_{k-1}$ and $\mathbb{P}(d(X_n, X) > \frac{1}{2^k}) < \frac{1}{2^k}$. Then $\sum_{k=1}^{\infty} \mathbb{P}(d(X_{n_k}, X) > \frac{1}{2^k}) < \infty$. By the first Borel Cantelli lemma $\mathbb{P}(d(X_{n_k}, X) > \frac{1}{2^k}$ *i.o.*$) = 0$. Hence $X_{n_k} \xrightarrow{as} X$ $\forall \omega \in A^c$, $\mathbb{P}(A^c) = 1$. ■

15.1.1 Completeness of L^p spaces

Definition 15.9 *Let (X, d) be a metric space. A sequence $\{X_n\} \subset \chi$ is Cauchy if $\forall \epsilon > 0$ $\exists n$ s.t. $\forall n, m > N$, $d(X_n, X_m) < \epsilon$.*

Example: Let $X = \mathbb{Q}$, $d(x, y) = |x - y|$. Then $x_n = \left(1 + \frac{1}{n}\right)^n$ but $x_n \to e \notin \mathbb{Q}$. Therefore $\mathbb{Q}$ is not complete.

Claim 15.10 *If x_n is Cauchy and a subsequence converges to x, then the whole sequence converges to x.*

Lemma 15.11 *L^p spaces are complete.*

15.2 Strong Law of Large Numbers

Theorem 15.12 *(Kolmogorov's maximal inequality) Let $X_1, \ldots, X_n$ be independent random variables with mean 0 and finite variance. Let $S_k = \sum_{i=1}^{k} X_i$, $k = 1, \ldots, n$; then*

$$\mathbb{P}(\max_{k=1,\ldots,n} |S_k > \epsilon) \leq \frac{Var(S_n)}{\epsilon^2}$$

Theorem 15.13 *Let $X_1, \ldots, X_n$ be independent random variables with mean 0 and bounded variance. If $\sum_{i=1}^{n} Var(X_i) = \sum_{i=1}^{n} \sigma_i^2 < \infty$ $\forall n$, then $\exists S_\infty$ s.t. $S_n \to S_\infty$ almost surely and in L^2 with $\mathbb{E}[S_\infty^2] = \sum_{i=1}^{\infty} \sigma_i^2$.*

Proof: L^2 convergence follows from completeness of the space L^2.
For almost surely convergence: S_n is almost surely Cauchy if $\sup_{p,q>n} |S_p - S_q| \xrightarrow{as} 0$ $\forall n$. This occurs if $\mathbb{P}(\sup_{p,q>n} |S_p - S_q| > \epsilon) \to 0 \forall \epsilon$ by the homework. ■