Confidence Bands for Cumulative Distribution Functions of Continuous Random Variables

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Previously suggested methods for constructing confidence bands for cumulative distribution functions have been based on the classical Kolmogorov-Smirnov test for an empirical distribution function. This paper gives a method based on maximum likelihood estimation of the parameters. The method is described for a general continuous distribution. Detailed results are given for a location-scale parameter model, which includes the normal and extreme-value distributions as special cases. Results are also given for the related lognormal and Weibull distributions. The formulas derived for these distributions give a band with exact confidence coefficient. A chi-squared approximation, which avoids the use of special tables, is also described. An example is used to compare the resulting bands with those obtained by previously published methods.

KEY WORDS: Confidence band; Cumulative distribution function; Maximum likelihood estimator; Location-scale parameter models; Normal distribution; Lognormal distribution; Extreme-value distribution; Weibull distribution.

1. INTRODUCTION

The problem of estimating the unknown cumulative distribution function (cdf) of a continuous random variable often occurs in reliability studies and life testing. A typical case arises in strength tests of a given material. A small number of samples is taken and the stress at which each sample breaks is then measured. The problem is to use these observations to obtain estimates of the proportion of samples in a large batch that will break at a given stress, for a whole range of stress values. This problem can be formulated as one where an unknown cdf has to be estimated. It may not always be enough to give an estimate of the cdf alone. In the strength-test example, it may also be important to know that estimates of the proportion of samples breaking at different stress values are not in error by more than some specified amount. Some indication is thus needed of the accuracy of the estimated cdf. One approach is to construct a confidence band that will contain, with a high confidence level, the entire unknown cdf. This is the problem considered in this paper.

Several methods have been described for constructing confidence bands for cdf's. Steck (1971) gives very general confidence bands by finding the probability that the empirical cdf lies between two arbitrary cdf's. Most methods, however, are based on the Kolmogorov-Smirnov test. Bradley (1968) gives a general version using this approach, while Breth (1978) shows how Bayesian confidence bands can be constructed with this technique. The most attractive solutions using this method are those of Kanofsky and Srinivasan (1972) for the normal distribution, and Srinivasan and Wharton (1975) for the Weibull distribution.

A weakness of methods which make use of the Kolmogorov-Smirnov test is that they give rise to a band which is of constant (vertical) width. This makes the band unnecessarily broad in the tails. In fact, part of such a band will have ordinate values that are greater than one, while part will have ordinate values that are negative. Obviously these are values which no cdf can take. Kanofsky and Srinivasan recognize this problem and overcome it in the case of the normal distribution by exploiting special properties of the distribution to cut away certain portions of the initial band without altering the confidence coefficient of the band. The full method requires the initial band to be constructed using special tables, and then, again using special tables, four further curves to be calculated that are used to "whittle down" the initial band. Srinivasan and Wharton apply the same method to the Weibull distribution, but it should be pointed out that there appear to be numerical errors in the table used for calculating the curves which are used for whittling down the initial band. Consequently the table does not always give curves that intersect the initial band at the points claimed for them.

The method suggested in this paper is based on a different idea. It can be applied in principle to any continuous distribution that is dependent on a number of unknown parameters. First, a confidence region is constructed for the unknown parameters. A confidence band is then obtained by seeing how the cdf changes as the parameters are varied within the confidence region. The method has the merit of giving a band directly, without the need to resort to any whittling-down procedure. In particular, the method automatically gives a band whose ordinate values lie between 0 and 1.

To illustrate the method, explicit formulas are given for confidence bands for the cdf of a location-scale parameter model. Bands for the normal and extreme-value distributions are given as special cases. In both these cases, the method has the practical advantage that the bands are given explicitly in terms of the cdf of a known distribution and a single constant. The constant can be approximated by the chi-squared quantile with little error even for small sample sizes. Tables are given that provide the exact confidence coefficient if this approximation is used and thus enable the user to assess the adequacy of the approximation. As an alternative, we include tables of the exact values for the constant when it is required to calculate confidence bands with a specified exact confidence coefficient. Confidence bands for the lognormal and Weibull distributions follow very simply from the normal and extreme-value cases, respectively.

Numerical examples are given that provide a comparison between the suggested new method and the earlier methods of Kanofsky and Srinivasan (1972) and Srinivasan and Wharton (1975).

The general method is described in Section 2, and the particular cases in Section 3. The numerical examples follow in Section 4.

2. METHOD

Let X be a continuous random variable with cumulative distribution function $F(x, \theta)$ dependent on a vector θ of k unknown parameters. A random sample of size n is drawn from which it is desired to calculate a confidence band for the cdf. (In direct analogy to the standard definition of a confidence region, a $100(1-\alpha)$ percent confidence band, B, for the graph of the cdf of X, $y = F(x, \theta)$, is defined as a region in the (x, y) plane, in which the graph of the unknown true cdf will entirely lie with probability $1-\alpha$.) The method suggested below produces a band by first obtaining a confidence region for θ. The band is then derived from this region.

Suppose first that a $100(1-\alpha)$ percent confidence region R has already been constructed in the parameter space for the unknown parameter vector. Then, considering the graph $y = F(x, \theta)$ in the (x, y) plane we look at the way this varies as θ varies in R. These varying graphs will sweep out an S-shaped region, B. As the true value of θ lies in R with probability $1-\alpha$ this means that the probability is at least $1-\alpha$ that one of the graphs used to sweep out the region B is the unknown true cdf of X. Thus B is a confidence band for $F(x, \theta)$ in which the true cdf will entirely lie, with probability at least $1-\alpha$. It is conceivable that there may be values of θ outside R which give rise to cdf's lying entirely within the band B, thus increasing the confidence coefficient and making B conservative. In general, it does not appear possible to give simple conditions on R which ensure that B is not conservative in this sense, but it turns out that the bands for the location-scale parameter models given below are not conservative. This follows from special properties of these distributions.

A method for determining the upper and lower envelopes of the confidence band consists of taking the pth quantile value x_p, defined by

$$P[X \le x_p] = F(x_p, \theta) = p, \tag{2.1}$$

and looking at how x_p varies, for fixed p, as θ varies in R. Mathematically, the problem is thus to find the maximum and minimum values of x_p given by equation (2.1), subject to the condition $\theta \in R$. The method depends on finding R, a confidence region for θ, as its first step.

Our suggested choice of R requires the maximum likelihood estimator, $\hat{\theta}$, of θ to be determined first. Use is then made of the well-known result, given for example in Kendall and Stuart (1961), that $\hat{\theta}$ has asymptotically a multivariate normal distribution with mean θ and variance-covariance matrix $(I(\theta))^{-1}$, where $I(\theta)$ is the Fisher information matrix, $-E(\partial^2 \ln L/\partial\theta_i\, \partial\theta_j)$, and L is the likelihood function. This means that

$$Q(\theta) = (\hat{\theta} - \theta)^T I(\theta)(\hat{\theta} - \theta) \tag{2.2}$$

is asymptotically a chi-squared variable with k degrees of freedom.

Let γ be the value for which

$$P[Q(\theta) \le \gamma] = 1 - \alpha. \tag{2.3}$$

A confidence region R for the unknown θ can be constructed by taking all θ satisfying $Q \le \gamma$. This yields a region R, which is asymptotically ellipsoidal in shape, whose boundary is given by all θ satisfying

$$(\hat{\theta} - \theta)^T I(\theta)(\hat{\theta} - \theta) = \gamma. \tag{2.4}$$

If γ is set equal to the chi-squared quantile $\chi_k^2(\alpha)$ defined by $P(\chi^2 \le \chi_k^2(\alpha))$ (where χ^2 is a chi-squared variable with k degrees of freedom), the confidence coefficient of the region will tend to $1 - \alpha$ with increasing sample size. The precise value of the confidence coefficient can be obtained by evaluating the probability (2.3) with γ set equal to $\chi_k^2(\alpha)$, either by quadrature or by Monte Carlo simulation. For the specific distributions considered in the next section $\chi_k^2(\alpha)$ turns out to be a good approximation to γ for most sample sizes. Alternatively, γ can be calculated to obtain a band with a prescribed exact confidence coefficient.

It should be mentioned that other quantities such as $Q_1(\theta) = (\hat{\theta} - \theta)^T I(\hat{\theta})(\hat{\theta} - \theta)$ or $Q_2(\theta) = -2 \ln[L(\theta)/L(\hat{\theta})]$, where $L(\theta)$ is the likelihood function, could also have been used instead of $Q(\theta)$ as they are also asymptotically χ_k^2 variables. For small samples $Q_2(\theta)$ is usually a better approximation to χ_k^2 than $Q(\theta)$ or $Q_1(\theta)$. However $Q_2(\theta)$ is a more complicated expression than $Q(\theta)$ or $Q_1(\theta)$ and its use would increase the labor of calculating the band. An advantage of working with $Q(\theta)$ is that $I(\theta)$ is the same for all samples of a given size n and hence one can solve once and for all for the confidence band.

3. APPLICATIONS TO LOCATION-SCALE PARAMETER AND RELATED MODELS

3.1 The General Location-Scale Parameter Model

We now apply the method of the previous section for constructing confidence bands to a general location-scale parameter model. The normal and extreme-value models are then considered as special cases. The related lognormal and Weibull models follow from these. The reader requiring practical results for any of these cases without details of derivation may wish to consult the appropriate subsection directly.

We start with a known continuous distribution with cdf $F(\xi)$ say, and probability density function (pdf) $f(\xi)$, and consider the random variable X with cdf $F[(x - \mu)/\sigma]$ where μ and σ are the unknown location and scale parameters.

For a random sample $x_1, x_2, \ldots, x_n$ the maximum likelihood estimates $\hat{\mu}$, $\hat{\sigma}$ can be found in the standard way by solving the likelihood equations. For the above location-scale model, if we write $g(\xi) = \ln f(\xi)$ and $g'(\xi) = dg(\xi)/d\xi$, these equations are (see, e.g., Kendall and Stuart 1961)

$$\sum g'(\hat{\xi}_i) = 0 \qquad \sum \hat{\xi}_i g'(\hat{\xi}_i) + n = 0 \tag{3.1}$$

where $\hat{\xi}_i = (x_i - \hat{\mu})/\hat{\sigma}$.

The confidence region R for the unknown μ and σ can now be constructed. For the location-scale model the information matrix is (see for example Kendall and Stuart 1961) of the form

$$I(\mu, \sigma) = n\sigma^{-2}\begin{pmatrix} k_0 & -k_1 \\ -k_1 & k_2 \end{pmatrix} \tag{3.2}$$

where k_0, k_1, k_2 are constants independent of μ and σ. Equation (2.4), giving the boundary of R thus reduces to

$$C(\mu, \sigma) = nk_0(\hat{\mu} - \mu)^2/\sigma^2$$
$$-2nk_1(\hat{\mu} - \mu)(\hat{\sigma} - \sigma)/\sigma^2 + nk_2(\hat{\sigma} - \sigma)^2/\sigma^2 = \gamma. \tag{3.3}$$

As was shown in Section 2, γ can be approximated by a χ^2 percentile. The adequacy of this approximation will be examined for particular distributions later. Note that, to find exact values of γ requires calculation of the distribution of the middle expression of (3.3), where this is regarded as a random variable dependent on the distributions of $\hat{\mu}$ and $\hat{\sigma}$. This variable can be written as

$$Q = nk_0 M^2 - 2nk_1 MS + nk_2 S^2, \tag{3.4}$$

where $M = (\hat{\mu} - \mu)/\sigma$ and $S = (\hat{\sigma} - \sigma)/\sigma$ and μ and σ are the unknown parameter values. Now the likelihood equations (3.1) can also be rewritten in terms of M and S using the substitution $\hat{\xi}_i = (\xi_i - M)/(1 + S)$ where $\xi_i = (x_i - \mu)/\sigma$. Since the ξ_i have a distribution independent of μ and σ, it follows that M and S and hence Q are also independent of μ and σ. This property greatly simplifies tabulation of exact values for γ.

Equation (3.3) can be rearranged by multiplying by σ^2. If $\sigma > 0$ the solutions are clearly unaltered. When multiplied by σ^2 (3.3) becomes a quadratic in μ and σ. The boundary of R is thus an ellipse when the discriminant is negative. This obtains when

$$\gamma/n < (k_0 k_2 - k_1^2)/k_0. \tag{3.5}$$

This condition turns out not to be stringent in practice and will be assumed to hold in what follows.

We now consider how quantiles of the distribution vary as (μ, σ) varies in R. The pth quantile is found from the equation $(x_p - \mu)/\sigma = F^{-1}(p) = a$, say. This gives x_p explicitly as

$$x_p = \mu + a\sigma. \tag{3.6}$$

In the (μ, σ) plane, if p and hence a is fixed, equation (3.6) shows that the values of (μ, σ) which give a constant x_p lie along a straight line with slope $-a^{-1}$ and intercept with the μ axis equal to x_p. A family of parallel lines is thus generated by varying x_p. Still with p and a fixed, we look at how x_p varies if (μ, σ) is constrained to be in R. Clearly the smallest and largest values obtainable, $\hat{x}_p(\min)$ and $\hat{x}_p(\max)$, correspond to two parallel tangents to the ellipse R (see Figure 1). This pair of values thus determines the limits of the band B at a given value of p.

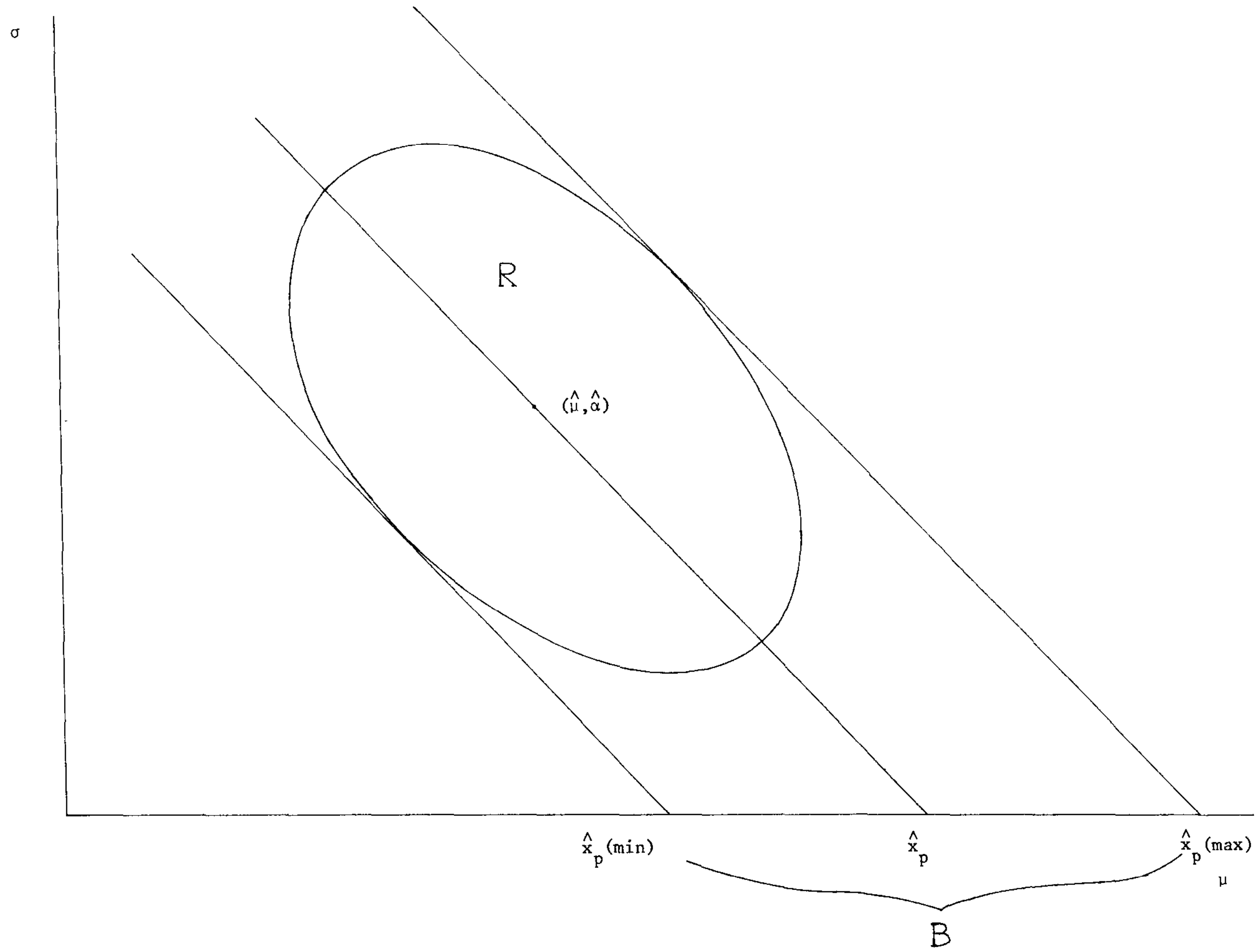

Figure 1. *Confidence Region for Parameters of a Location-Scale Parameter Model and Band B for the pth Quantile*

Explicit formulas for $\hat{x}_p(\min)$ and $\hat{x}_p(\max)$ can be found as follows. Since the maximum and minimum of x_p occur for (μ, σ) lying on the boundary of R, the method of Lagrangian multipliers can be used to solve the problem. The Lagrangian is $x_p + \eta C(\mu, \sigma)$, where x_p is given by (3.6), $C(\mu, \sigma)$ by (3.3) and η is the Lagrange multiplier. The maximum and minimum are found by setting the partial derivatives with respect to μ and σ equal to zero:

$$\frac{\partial x_p}{\partial \mu} + \eta \frac{\partial C}{\partial \mu} = 0, \qquad \frac{\partial x_p}{\partial \sigma} + \eta \frac{\partial C}{\partial \sigma} = 0.$$

The multiplier η can be eliminated by division

$$\frac{\partial x_p}{\partial \mu} \bigg/ \frac{\partial x_p}{\partial \sigma} = \frac{\partial C}{\partial \mu} \bigg/ \frac{\partial C}{\partial \sigma}. \tag{3.7}$$

Equation (3.7) can be expressed in terms of the variables M and S as defined in (3.4) and quadratic terms eliminated by using equation (3.3), rewritten in terms of M and S to give

$$(k_1 + ak_0)M = \gamma/n + (k_2 + ak_1)S. \tag{3.8}$$

This expression can be used to eliminate M from (3.3) to give a quadratic in S

$$b(k_0 k_2 - k_1^2)S^2 + 2(\gamma/n)(k_0 k_2 - k_1^2)S + k_0(\gamma/n)^2 - (k_1 + ak_0)^2\gamma/n = 0,$$

where $b = k_2 + 2k_1 a + k_0 a^2$. The roots of this equation are

$$S = [-1 \pm (k_1 + ak_0)(nb/\gamma - 1)^{1/2} \times (k_0 k_2 - k_1^2)^{-1/2}]\gamma/(nb). \tag{3.9}$$

These roots are real so long as $b > \gamma/n$. The smallest value b can take as a varies is $(k_0 k_2 - k_1^2)/k_0$, so this condition, that $b > \gamma/n$ for all values of a, reduces to the very one given earlier in (3.5) that R should be an ellipse.

In terms of M and S, equation (3.6) for the pth percentile x_p is

$$x_p = \hat{x}_p - \hat{\sigma}(M + aS)(1 + S)^{-1} \tag{3.10}$$

where $\hat{x}_p = \hat{\mu} + a\hat{\sigma}$.

The value of S obtained by taking the negative square root in (3.9) and the corresponding value of M from (3.8) are then substituted into (3.10) to give the largest value $\hat{x}_p(\max)$. The positive value of the square root gives the smallest value $\hat{x}_p(\min)$. These formulas reduce to

$$\hat{x}_p(\max) = \hat{x}_p + \hat{\sigma}b(N - k_1 - ak_0)^{-1}$$
$$\hat{x}_p(\min) = \hat{x}_p - \hat{\sigma}b(N + k_1 + ak_0)^{-1} \quad (3.11)$$

where

$$N = [(k_0 k_2 - k_1^2)(nb/\gamma - 1)]^{1/2}$$

and

$$b = k_2 + 2k_1 a + k_0 a^2.$$

Alternatively the band can be specified in terms of its upper and lower limits at each x value. This can be done by inverting (3.11) so as to give p in terms of x. This yields upper and lower limits of the band as

$$\hat{F}_{\max}(x) = F(\hat{\xi} + h)$$
$$\hat{F}_{\min}(x) = F(\hat{\xi} - h) \quad (3.12)$$

where

$$h = \{\gamma n^{-1} k_0^{-1}[1 + (k_0\hat{\xi} + k_1)^2 \times (k_0 k_2 - k_1^2)^{-1}]\}^{1/2} \quad \text{and} \quad \hat{\xi} = (x - \hat{\mu})/\hat{\sigma}.$$

That the band is exact, and not conservative in the sense of Section 2, can be shown as follows. The only values of (μ, σ) that give rise to a cdf lying entirely within the band B are those that give x_p values which lie within the limits $\hat{x}_p(\min)$ and $\hat{x}_p(\max)$ for all p, $0 \le p \le 1$. But, as p varies, $0 \le p \le 1$, the pair of parallel tangents giving $\hat{x}_p(\min)$ and $\hat{x}_p(\max)$ rotate about R in such a way that the slopes will take all values between $-\infty$ and ∞. This means, as R is an ellipse and hence convex, that the (μ, σ) values which lie simultaneously between all tangent pairs consist precisely of those in the region R. This shows that the band B is not conservative, as no (μ, σ) value outside R is possible that will give a cdf lying entirely within the band B.

3.2 Normal Distribution

The results of the previous subsection specialize immediately to give a confidence band for the cdf of the normal distribution with unknown mean μ and variance σ^2. The maximum likelihood estimates of μ and σ are well known to be

$$\hat{\mu} = \bar{x} \quad \text{and} \quad \hat{\sigma} = \{\textstyle\sum (x_i - \bar{x})^2/n\}^{1/2}.$$

The constants in the Fisher information matrix $I(\mu, \sigma)$ are $k_0 = 1$, $k_1 = 0$, and $k_2 = 2$ (see Bury 1975) and, in the expression for the pth quantile (equation 3.6), $a = z_p$, where z_p is the pth quantile of the standardized normal distribution. Equations (3.11) giving the limits of the confidence band with coefficient $1 - \alpha$ thus reduce to

$$\hat{x}_p(\max) = \hat{x}_p + \hat{\sigma}b(N - z_p)^{-1}$$
$$\hat{x}_p(\min) = \hat{x}_p - \hat{\sigma}b(N + z_p)^{-1} \quad (3.13)$$

where $b = 2 + z_p^2$, $N = [2nb/\gamma - 2]^{1/2}$, and $\hat{x}_p$ is the pth quantile estimator

$$\hat{x}_p = \hat{\mu} + \hat{\sigma}z_p. \quad (3.14)$$

Here γ is the αth quantile of Q as defined in (3.3). The discussion following equation (3.4) shows that the distribution of Q is independent of μ and σ. Thus γ depends only on α and the sample size. Selected values are given in Table 1. If the approximation $\chi_2^2(\alpha)$ is used for γ, exact values of the confidence coefficient are given in Table 2.

The tables were obtained as follows. The variable Q is

$$Q = n(\hat{\mu} - \mu)^2/\sigma^2 + 2n(\hat{\sigma} - \sigma)^2/\sigma^2 = q_1 + q_2 \quad \text{(say)}.$$

Now q_1 is well known to be a chi-squared variable with one degree of freedom. The distribution of q_2 is more complicated. It is well-known that $n\hat{\sigma}^2/\sigma^2$ is a chi-squared random variable with $(n - 1)$ degrees of

Table 1. *Exact Values of γ for the Normal Distribution, Equation* (3.13)

Sample Size	1−α 0.800	0.850	0.900	0.950	0.990
5	3.544	4.138	4.961	6.350	9.750
6	3.485	4.078	4.903	6.300	9.636
7	3.443	4.035	4.861	6.260	9.567
8	3.413	4.003	4.829	6.229	9.520
9	3.390	3.979	4.804	6.204	9.484
10	3.372	3.960	4.783	6.183	9.457
12	3.345	3.931	4.753	6.152	9.416
14	3.326	3.911	4.732	6.130	9.387
16	3.312	3.986	4.716	6.113	9.365
18	3.301	3.884	4.703	6.099	9.348
20	3.293	3.875	4.693	6.089	9.335
25	3.278	3.858	4.675	6.069	9.310
30	3.268	3.848	4.664	6.056	9.294
35	3.261	3.840	4.655	6.047	9.282
40	3.255	3.834	4.649	6.040	9.274
50	3.248	3.826	4.640	6.031	9.262
60	3.243	3.821	4.634	6.024	9.253
80	3.237	3.814	4.627	6.016	9.244
100	3.233	3.810	4.623	6.011	9.238
∞	3.219	3.794	4.605	5.991	9.210

Table 2. *Exact Confidence Coefficient Corresponding to a $\chi_2^2(\alpha)$ Value Used as an Approximation to γ in the Normal Distribution Case, Equation* (3.13)

	1−α				
Sample Size	0.800	0.850	0.900	0.950	0.990
5	0.766	0.823	0.881	0.940	0.987
6	0.773	0.828	0.885	0.942	0.988
7	0.777	0.831	0.887	0.943	0.988
8	0.780	0.834	0.888	0.944	0.988
9	0.783	0.836	0.890	0.944	0.989
10	0.785	0.837	0.891	0.945	0.989
12	0.787	0.840	0.892	0.946	0.989
14	0.789	0.841	0.894	0.946	0.989
16	0.791	0.842	0.894	0.947	0.989
18	0.792	0.843	0.895	0.947	0.989
20	0.793	0.844	0.896	0.948	0.989
25	0.794	0.845	0.896	0.948	0.989
30	0.795	0.846	0.897	0.948	0.990
35	0.796	0.847	0.897	0.949	0.990
40	0.796	0.847	0.898	0.949	0.990
50	0.797	0.848	0.898	0.949	0.990
60	0.798	0.848	0.899	0.949	0.990
80	0.798	0.848	0.899	0.949	0.990
100	0.799	0.849	0.899	0.949	0.990

freedom and that the pdf of q_2 can be written in terms of the pdf of $\hat{\sigma}$, which in turn can be expressed in terms of the pdf of $n\hat{\sigma}^2/\sigma^2$. Moreover, since $\hat{\mu}$ and $\hat{\sigma}^2$ are independent, q_1 and q_2 are also independent. Thus the distribution of $q_1 + q_2$ is the convolution of the distributions of q_1 and q_2. The convolution was evaluated numerically using a standard integration program (NAG routine D01 AGF). Table 1 was obtained by varying γ iteratively until the confidence coefficient was equal to preselected values. Notice that the condition (3.5) for N to be real reduces to $\gamma < 2n$ and this is satisfied by all values of γ in Table 1. Table 2 was obtained simply by evaluating the confidence coefficient at the selected approximate values of γ.

3.3 The Lognormal Distribution

The probability density function $f(v)$ of a lognormally distributed random variable V is given by

$$f(v) = \begin{cases} (2\pi\sigma^2)^{-1/2}v^{-1}\exp\{-(2\sigma^2)^{-1} \times (\ln v - \mu)^2\}, & v > 0 \\ 0, & \text{otherwise.} \end{cases}$$

This is not itself a location-scale parameter model; however, if V is lognormally distributed $X = \ln V$ is normally distributed with mean μ and variance σ^2. Thus given a random sample drawn from the above lognormal distribution, a confidence band for the cdf can be constructed using (3.13). The sample is transformed to normality by taking natural logarithms and then the limits of the band $\hat{x}_p(\max)$ and $\hat{x}_p(\min)$ are calculated from equation (3.13). As the log transformation is monotone the band can be transformed back by exponentiating these limits to give limits for v_p as

$$\hat{v}_p(\max) = \exp\{\hat{x}_p(\max)\}$$
$$\hat{v}_p(\min) = \exp\{\hat{x}_p(\min)\}.$$

3.4 The Extreme-Value Distribution

The cdf of a random variable X that has the extreme-value distribution with unknown location parameter μ and scale parameter σ is

$$F(x) = \exp\{-\exp[-(x-\mu)/\sigma]\} \qquad -\infty < x < \infty. \quad (3.15)$$

The notation here follows that of Bury (1975). Note that some authors write the second exponential in (3.15) with a positive sign instead.

The maximum likelihood estimates $\hat{\mu}$ and $\hat{\sigma}$ of μ and σ cannot be given explicitly, but can be calculated from the likelihood equations by iterative methods. For details see, for example, Bury (1975). The constants in the Fisher information matrix $I(\mu, \sigma)$ in this case are $k_0 = 1$, $k_1 = 1 - c$, and $k_2 = \pi^2/6 + k_1^2$, where $c = 0.577216$ is Euler's constant (see Bury 1975). In the expression for the pth quantile (equation 3.6), $a = -\ln\ln(1/p)$. Equations (3.11) giving the limits of a confidence band with coefficient $1 - \alpha$ thus reduce to

$$\begin{aligned} \hat{x}_p(\max) &= \hat{x}_p + \hat{\sigma}b(N - k_1 - a)^{-1} \\ \hat{x}_p(\min) &= \hat{x}_p - \hat{\sigma}b(N + k_1 + a)^{-1} \end{aligned} \quad (3.16)$$

where $b = k_2 + 2k_1a + a^2$, $N = [\pi^2(nb/\gamma - 1)/6]^{1/2}$ and $\hat{x}_p$ is the pth quantile estimator

$$\hat{x}_p = \hat{\mu} + \hat{\sigma}a. \quad (3.17)$$

As in the normal model, γ depends only on α and the sample size. Selected values are given in Table 3. Table 4 gives the actual value of the confidence coefficient when γ is approximated by the chi-squared quantile $\chi_2^2(\alpha)$. The tables were obtained as follows. Calculation of γ requires evaluation of the distribution of the random variable Q of equation (3.3). Unlike the normal, this distribution is difficult to evaluate, even numerically. Instead it has been estimated by Monte Carlo methods. For each sample size the values of γ are based on 1,000,000 pseudo-random samples. Table 3 is believed to be accurate as far as the quoted

Table 3. *Exact Values of γ for the Extreme Value Case, Equation* (3.16)

Sample Size	0.80	0.85	1−α 0.90	0.95	0.99
5	3.70	4.34	5.27	7.08	*
6	3.62	4.25	5.17	6.89	*
7	3.57	4.19	5.10	6.76	11.48
8	3.53	4.16	5.05	6.69	11.26
9	3.50	4.12	5.01	6.61	11.00
10	3.47	4.08	4.97	6.55	10.80
12	3.44	4.05	4.92	6.47	10.55
14	3.40	4.01	4.88	6.40	10.34
16	3.38	3.98	4.84	6.34	10.21
18	3.37	3.97	4.83	6.33	10.10
20	3.36	3.96	4.81	6.30	10.04
25	3.33	3.93	4.77	6.25	9.89
30	3.31	3.91	4.74	6.20	9.74
35	3.30	3.88	4.72	6.16	9.64
40	3.29	3.87	4.70	6.14	9.61
50	3.26	3.85	4.68	6.11	9.52
60	3.26	3.85	4.67	6.08	9.44
70	3.25	3.84	4.66	6.07	9.48
80	3.25	3.83	4.65	6.07	9.38
100	3.24	3.82	4.64	6.04	9.34
∞	3.219	3.794	4.605	5.991	9.210

* The method fails if $\gamma > n\pi^2/6$ (see text).

Table 4. *Exact Confidence Coefficient Corresponding to a $\chi_2^2(\alpha)$ Value Used as an Approximation to γ in the Extreme-Value Case, Equation* (3.16)

Sample Size	0.800	0.850	1−α 0.900	0.950	0.990
5	0.752	0.809	0.867	0.925	*
6	0.759	0.815	0.872	0.929	*
7	0.765	0.820	0.875	0.932	0.979
8	0.768	0.823	0.878	0.933	0.980
9	0.771	0.826	0.880	0.935	0.982
10	0.774	0.828	0.882	0.936	0.982
12	0.778	0.831	0.884	0.938	0.984
14	0.781	0.834	0.887	0.940	0.984
16	0.784	0.836	0.888	0.941	0.985
18	0.786	0.837	0.889	0.942	0.986
20	0.786	0.838	0.890	0.943	0.986
25	0.789	0.840	0.892	0.944	0.987
30	0.791	0.842	0.893	0.945	0.987
35	0.792	0.843	0.894	0.946	0.988
40	0.793	0.844	0.895	0.946	0.988
45	0.795	0.846	0.896	0.947	0.988
50	0.796	0.846	0.897	0.948	0.989
60	0.797	0.847	0.897	0.948	0.989
80	0.796	0.847	0.898	0.948	0.989
100	0.798	0.848	0.898	0.949	0.989

* The method fails for these values (see text).

third decimal place. In Table 4 the values are produced by interpolation, and the error is believed to be at most $\pm.05$. The second decimal place is included for guidance only. The condition for N to be real is that $\gamma < n\pi^2/6$. The calculated values of γ for sample sizes 5 and 6 for 99 percent confidence bands exceed $n\pi^2/6$ and have not been quoted in Table 4.

3.5 The Weibull Distribution

The pdf $f(w)$ of the Weibull distribution is given by

$$f(w) = \begin{cases} \alpha\beta^{-\alpha}w^{\alpha-1}\exp\{-(w/\beta)^\alpha\} & w > 0 \\ 0 & \text{otherwise} \end{cases} \tag{3.18}$$

This is not a location-scale parameter model but if W is a Weibull-distributed random variable then $X = -\ln W$ has the extreme value distribution with cdf (3.15). The parameters of the two distributions are related by the equations $\alpha = \sigma^{-1}$ and $\beta = \exp(-\mu)$. Thus a random sample from the Weibull distribution may be transformed by taking natural logarithms and changing the sign, to give a random sample for an extreme value distribution. A confidence band for the extreme value cdf of the transformed sample can be calculated from (3.16), to give values $\hat{x}_p(\max)$ and $\hat{x}_p(\min)$ for any p.

A band for the Weibull cdf of the original sample is obtained by noting that the pth quantile w_p of the Weibull distribution can be obtained from the $(1 - p)$th quantile x_{1-p} of the extreme value distribution from the equation

$$w_p = \exp(-x_{1-p}).$$

As the log transformation is monotone, the limits $\hat{x}_{1-p}(\max)$ and $\hat{x}_{1-p}(\min)$ can be transformed back by exponentiating the limits after changing the sign to give limits for w_p as

$$\begin{aligned} \hat{w}_p(\max) &= \exp\{-\hat{x}_{1-p}(\min)\} \\ \hat{w}_p(\min) &= \exp\{-\hat{x}_{1-p}(\max)\}. \end{aligned} \tag{3.19}$$

4. NUMERICAL EXAMPLES

To illustrate the use of the formulas of Section 3 we construct confidence bands for the cdf's for the log-

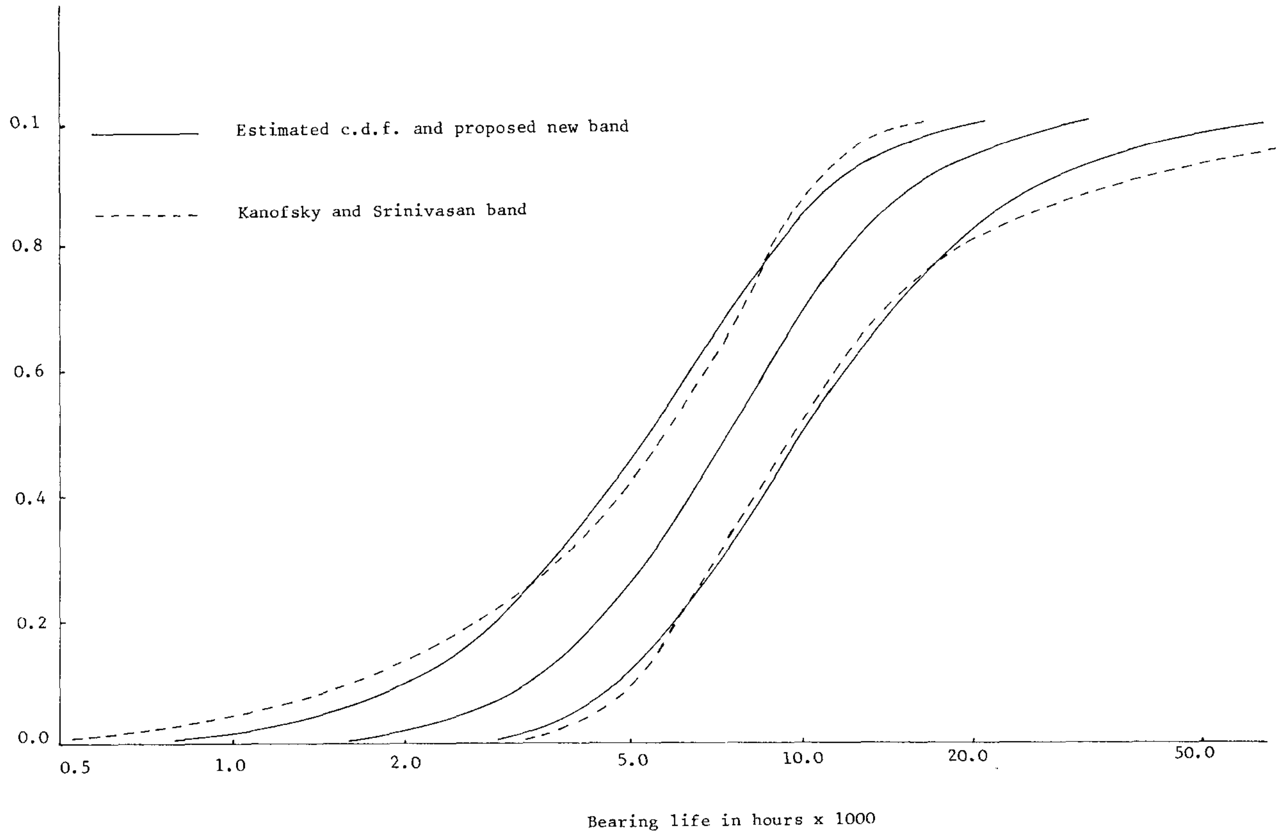

Figure 2. *Estimated Distribution Function and* 90% *Confidence Bands for Normal Model and Kanofsky and Srinivasan Band*

normal and Weibull distributions for a particular set of data. It is not the intention to make any formal comparison of the fits of the different models; the aim is merely to illustrate the ease with which bands can be calculated.

Schafer and Angus (1979) gave the operational lives (in hours) of 20 bearings. These are 2398, 2812, 3113, 3212, 3523, 5236, 6215, 6278, 7725, 8604, 9003, 9350, 9460, 11584, 11825, 12628, 12888, 13431, 14266, 17809.

In order to calculate the cdf and the confidence bands using a lognormal model the data are first transformed by taking natural logarithms. The maximum likelihood estimates of the mean and standard deviation of these transformed data are respectively $\hat{\mu} = 8.8984$ and $\hat{\sigma} = .5970$. The estimated cdf is calculated from (3.14) for a range of values of p between 0 and 1. Equation (3.13) enables upper and lower limits to be placed on these estimates. For example, when $\alpha = .1$ and $p = .90$, $a = z_p = 1.28155$ and from equation (3.14) $\hat{x}_p = 9.6635$. Table 1 gives the exact value of γ for a 90 percent confidence band and sample size 20 of 4.693. The values of b and N for equation (3.13) are $b = 3.6424$ and $N = 5.3894$. Equation (3.13) gives $\hat{x}_p(\min) = 9.338$ and $\hat{x}_p(\max) = 10.193$. If the χ^2 approximation to γ is used (i.e. $\gamma \simeq \chi_2^2\ (0.1) = 4.605$), $N = 5.4441$. Equation (3.13) then also gives $\hat{x}_p(\min) = 9.340$ and $\hat{x}_p(\max) = 10.186$. The values of $\hat{x}_p$, $\hat{x}_p(\min)$, and $\hat{x}_p(\max)$ have to be transformed by exponentiating to give the required quantile $\hat{v}_p$ and limits for the 90 percent confidence band, $\hat{v}_p(\min)$ and $\hat{v}_p(\max)$ for the lognormal model. These values are $\hat{v}_p = 15{,}733$ with $\hat{v}_p(\min) = 11{,}356$ and $\hat{v}_p(\max) = 26{,}710$ when the exact value of γ is used and $\hat{v}_p(\min) = 11{,}386$ and $\hat{v}_p(\max) = 26{,}527$ when the approximate value of γ is used. Similar calculations were done for a range of values of p. Figure 2 is a diagram of the estimated cdf and the 90 percent confidence bands using the χ^2 approximation to γ. Using the exact value of γ makes no discernible difference to the diagram.

The use of (3.12) to calculate limits for the cdf $F(x)$ at a fixed value of x can also be illustrated with these data. For example, corresponding to a lifetime of $v = 10{,}000$ hours, $x = \ln(10{,}000) = 9.2103$ and ξ from equation (3.12) is .5225. With the appropriate values for k_0, k_1, and k_2, $h = .5115$; whence $\hat{F}_{\max} = F(1.034)$ and $\hat{F}_{\min} = F(0.011)$, F in this case being the cdf of the standard normal distribution. Thus $\hat{F}_{\max} = 0.849$ and $\hat{F}_{\min} = 0.504$. Since the log transformation is monotonic increasing these are the appropriate bands for the lognormal model at $v = 10{,}000$ hours.

For comparison, the method of Kanofsky and Srinivasan (1972) for constructing a confidence band has

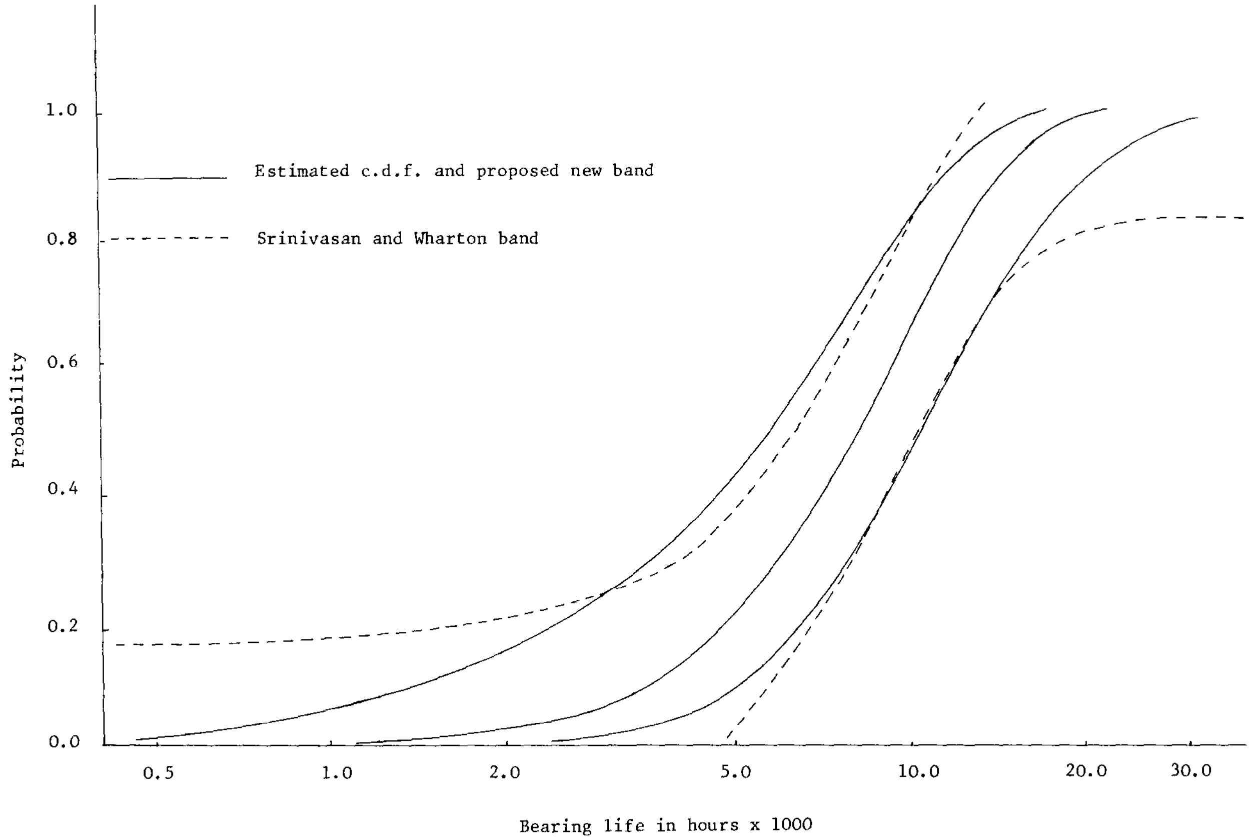

Figure 3. *Estimated Distribution Function and* 90% *Confidence Bands for Weibull Model and Srinivasan and Wharton Band*

also been used. This method defines the band in terms of upper and lower limits for the cdf F in the form

$$\hat{F}(\max) = \hat{F} + l_\alpha$$
$$\hat{F}(\min) = \hat{F} - l_\alpha \qquad (4.1)$$

where $\hat{F}$ is the estimated cdf. Now l_α is difficult to evaluate explicitly, but Kanofsky and Srinivasan (1972) have tabulated l_α for a wide range of values. In this example $l_{0.1} = .16$. This band is rather broad in the tails. Kanofsky and Srinivasan (1972) indicate how the tails can be whittled down using curves that intersect the band in (4.1) and it is this reduced band that is drawn in Figure 1. For this example it would appear that the bands obtained by the method of Kanofsky and Srinivasan are narrower in the central part of the cdf but wider in the tails.

For the fitting of the Weibull distribution it is again necessary to transform the data. The natural logarithms are taken and the sign is changed. Maximum likelihood estimates of the parameters are obtained by the iterative method suggested in Bury (1975). The values obtained are $\hat{\mu} = -9.1796$, $\hat{\sigma} = .4748$. These values enable the estimate of the cdf to be evaluated using (3.17) to work out the quantiles x_p for a range of values of p. Equations (3.16) are then used to evaluate the confidence band. As an example, when $\alpha = .1$ and $p = .90$, $1 - p = .10$ and $a = -.8340$. From equation (3.17) $\hat{x}_p = -9.5756$. From Table 3, the exact value of γ for a 90 percent confidence band and sample size 20 is 4.81. The values of b and N of equations (3.16) are respectively $b = 1.8141$ and $N = 3.2806$, giving $\hat{x}_{1-p}(\max) = -9.342$ and $\hat{x}_{1-p}(\min) = -9.876$. Alternatively, the χ^2 approximation to γ ($\gamma \simeq \chi_2^2(.1) = 4.605$) gives $N = 3.3638$, $\hat{x}_{1-p}(\max) = -9.348$ and $\hat{x}_{1-p}(\min) = -9.867$. The required quantile $\hat{w}_p$ and the limits for the confidence band $\hat{w}_p(\min)$ and $\hat{w}_p(\max)$ can now be obtained from equation (3.19). These values are $\hat{w}_p = 14{,}409$ with $\hat{w}_p(\min) = 11{,}411$ and $\hat{w}_p(\max) = 19{,}454$ when the exact value of γ is used and $\hat{w}_p(\min) = 11{,}470$ and $\hat{w}_p(\max) = 19{,}291$ when the approximate value of γ is used. Similar calculations were performed for a range of values of p. The band obtained is shown in Figure 3. As in the normal case, use of the approximate value of γ rather than the exact value makes no discernible difference to the figure.

Bands for the cdf $F(x)$ for a fixed value of x can be worked out from (3.12). For example, corresponding to a lifetime of $w = 10{,}000$ hours $x = -\ln(10{,}000) = -9.2103$ and $\xi = -.0647$. With the appropriate values of k_0, k_1, and k_2, $h = .4982$ so that $\hat{F}_{\max} =$

$F(.4335)$ and $\hat{F}_{\min} = F(-.5629)$. In this case $F(\xi) = \exp(-\exp[-\xi])$ is the standardized cdf of the extreme-value distribution corresponding to (3.15) with $\mu = 0$ and $\sigma = 1$. This gives $F(.4335) = .523$ and $F(-.5629) = .173$. The transformation from the extreme-value to the Weibull model is monotonic decreasing, so the appropriate limits for the confidence bands for the Weibull cdf at $w = 10{,}000$ hours are $1 - .523 = .477$ and $1 - .173 = .827$.

Srinivasan and Wharton (1975) gave a method of constructing a confidence band of the form (4.1). In this example, with significance level 0.1 $l_{0.1} = .17$; this band is also shown in Figure 3. They provide a suggested procedure for whittling down the tails, but Table 3 of their paper appears to be in error. The maximum likelihood estimates of the parameters α and β of the Weibull pdf (3.18) are respectively $\hat{\alpha} = 2.1060$ and $\hat{\beta} = 9697$. The initial lower limit for the confidence band for the cdf is then obtained by substituting these values into the Weibull cdf $F(w;\, \alpha,\, \beta)$ corresponding to the pdf (3.18) and subtracting $l_{0.1} = .17$. In the right tail this curve is replaced by a different Weibull cdf. These two curves are supposed to intersect at $w_2 = \hat{\beta}(x_2)^{1/2}$, where x_2 is obtained from tables provided by Srinivasan and Wharton. From these tables $x_2 = 2.9101$, so $w_2 = 16104$. The parameters for the cdf for the right tail are obtained by multiplying the maximum likelihood estimates by factors, also obtained from tables. These factors, using the notation of Srinivasan and Wharton, are $\alpha_1 = .4713$ and $\beta_1 = 1.2416$, which gives $\alpha = .9926$ and $\beta = 12040$. These values of α and β are substituted into $F(w;\, \alpha,\, \beta)$. Unfortunately $F(16104;\, .9926,\, 12040) = .7367$ whereas $F(16104;\, 2.1060,\, 9697) - .17 = .7755$ so that the two curves do not intersect at w_2. The narrower bands have not therefore been shown in Figure 3.

5. CONCLUDING COMMENTS

The method suggested in this paper for calculating confidence bands for a cdf of a continuous random variable appears to compare favorably with previous methods. For the normal, lognormal, extreme-value, and Weibull distributions the suggested methods give simple results. The equations given in Section 3 involve a certain amount of calculation, but this can readily be done either on a computer or on a programmable calculator, say. The method is appealing in that a single band is produced from the equations. There is no need to start from a band that is clearly too broad in the tails and whittle it down.

The general idea of working out such bands by first obtaining a confidence region for the parameters could be applied to any continuous random variable. Two things would have to be checked. First, the confidence region for the parameters we have suggested is based on an asymptotic result; the adequacy of the approximation would have to be checked for small sample sizes using methods like those used in this paper. Second, care would have to be taken to show that the bands obtained were exact bands with the stated confidence coefficient and not conservative bands. The point was discussed in Section 2, and Section 3 outlines how the point has to be checked in specific cases.

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