I have neither given nor received assistance on this exam.

Signature:

This exam is closed book and closed notes, except for one cheat sheet (one page, front). Calculators are permitted.

Show all work for full credit.
1. (8 points) Let $A, B \subset \Omega$ be two events such that $A \subset B$. Use the axioms of Probability to prove that $P(A) \leq P(B)$. 
2. (6 points) Let $A, B \subset \Omega$ be such that $A \subset B$, $P(A) > 0$ and $P(B) > 0$. Can the events $A$ and $B$ be independent? Clearly justify your answer.
3. (8 points) Consider a random variable $X$ and its (partially known) p.m.f.

$$p_X(x) = \begin{cases} 
? & \text{if } x = 0 \\
1/2 & \text{if } x = 1 \\
? & \text{if } x = 2 \\
1/10 & \text{if } x = 3 \\
0 & \text{if } x \notin \{0, 1, 2, 3\}
\end{cases}$$

Assume that $E(X) = 4/5$. Compute $V(X)$. 

4. (8 points) Let \( A, B \subset \Omega \) be such that \( P(A) = 0.4 \), \( P(B|A^c) = 0.1 \), and \( P(B^c|A) = 0.8 \). Compute \( P(B) \).
5. (10 points) Consider an urn with 3 red balls, 2 blue ball, and 1 white ball. You draw 2 balls from the urn without replacement.

- (1 points) Write the sample space \( \Omega \) for this experiment (assume that order matters here).
- (1 point) Are the simple events in \( \Omega \) all equally likely? Clearly justify your answer.
- (2 points) Consider the event \( A = 'none of the drawn balls is blue' \). List all the simple events of \( \Omega \) that are contained in \( A \).
- (6 points) Let \( B \) be the event \( B = 'at least 1 red ball is drawn' \). Compute \( P(B) \).
6. (12 points) Let \( X_1, \ldots, X_n \) iid \( \sim \mathcal{N}(\mu, \sigma^2) \). Let \( m \in \{0, 1, \ldots, n\} \). What is the probability that exactly \( m \) of the \( X \)'s are larger than 1? Your answer must be expressed as a function of \( \Phi \), where \( \Phi \) is the c.d.f. of the standard Normal distribution.
7. (10 points) Consider the random variable \( X \sim f_X \) with p.d.f.

\[
f_X(x) = k(1 - x)x^3 \mathbb{1}_{[0,1]}(x)
\]

for some \( k > 0 \).

- (3 points) State the name of the distribution of \( X \) and its parameters.
- (3 points) What is the value of \( k \)?
- (2 points) What is \( E(X) \)?
- (2 points) What is \( V(X) \)?
8. (12 points) Let $X \sim \text{Exponential}(\beta)$ and let $\alpha > 0$.

- (8 points) Derive a closed-form expression for $E(X^\alpha)$.
- (4 points) Use the expression that you derived to show that $V(X) = \beta^2$. 

9. (14 points) Consider the pair of random variables $(X_1, X_2) \sim p_{X_1,X_2}$ with joint p.m.f.

\[
p_{X_1,X_2}(x_1, x_2) = \begin{cases} 
0.2 & \text{if } (x_1, x_2) = (0, 0) \\
0.1 & \text{if } (x_1, x_2) = (1, 0) \\
0.1 & \text{if } (x_1, x_2) = (2, 0) \\
0.1 & \text{if } (x_1, x_2) = (0, 1) \\
0.3 & \text{if } (x_1, x_2) = (1, 1) \\
0 & \text{if } (x_1, x_2) = (2, 1) \\
0.05 & \text{if } (x_1, x_2) = (0, 2) \\
0.05 & \text{if } (x_1, x_2) = (1, 2) \\
0.1 & \text{if } (x_1, x_2) = (2, 2).
\end{cases}
\]

- (4 points) Compute the marginal p.m.f. of $X_2$.
- (4 points) Compute the marginal c.d.f. of $X_2$.
- (4 points) Compute the conditional p.m.f. of $X_1$ given $X_2 = 1$.
- (2 points) Are $X_1$ and $X_2$ independent?
10. (12 points) Consider the pair of random variables \((X_1, X_2) \sim f_{X_1,X_2}\) with joint p.d.f.

\[
f_{X_1,X_2}(x_1, x_2) = kx_1^3x_2^2 \mathbb{1}_{\{x_1 \geq 0 \land x_2 \geq 0 \land x_1 + x_2 \leq 1\}}(x_1, x_2)
\]

and \(k > 0\).

- (4 points) Are \(X_1\) and \(X_2\) independent?
- (8 points) Compute the value of \(k\).