Name:

I have neither given nor received assistance on this exam.

Signature:

This exam is closed book and closed notes, except for one cheat sheet (one page, front). Calculators are permitted.

Show all work for full credit.
1. (8 points) Let $A, B \subset \Omega$ be two events such that $A \subset B$. Use the axioms of Probability to prove that $P(A) \leq P(B)$.

**Solution:** We have $B = (B \cap A) \cup (B \cap A^c) = A \cup (B \cap A^c)$ and this is clearly a union of two disjoint events. It follows from the countable additivity axiom that $P(B) = P(A) + P(B \cap A^c) \geq P(A)$ since $P(B \cap A^c) \geq 0$. 
2. (6 points) Let \( A, B \subset \Omega \) be such that \( A \subset B \), \( P(A) > 0 \) and \( P(B) > 0 \). Can the events \( A \) and \( B \) be independent? Clearly justify your answer.

\textbf{Solution:} We have

\[
P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)}.
\]

Thus, if \( P(B) = 1 \), \( P(A \mid B) = P(A) \) and in this case the events \( A \) and \( B \) are independent.

Similarly, if we consider

\[
P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A)}{P(A)} = 1,
\]

it is easy to see once again that if \( P(B) = 1 \) then \( P(B \mid A) = P(B) \) and \( A \) and \( B \) are independent.
3. (8 points) Consider a random variable $X$ and its (partially known) p.m.f.

$$
p_X(x) = \begin{cases} 
? & \text{if } x = 0 \\
1/2 & \text{if } x = 1 \\
? & \text{if } x = 2 \\
1/10 & \text{if } x = 3 \\
0 & \text{if } x \notin \{0, 1, 2, 3\}
\end{cases}
$$

Assume that $E(X) = 4/5$. Compute $V(X)$.

**Solution:** We have

$$E(X) = \sum_{x \in \text{supp}(X)} xp_X(x) = 0 \cdot p_X(0) + 1 \cdot p_X(1) + 2p_X(2) + 3p_X(3).$$

In numbers,

$$\frac{4}{5} = 0 \cdot p_X(0) + 1 \cdot \frac{1}{2} \cdot p_X(1) + 2p_X(2) + 3 \cdot \frac{1}{10} = \frac{1}{2} \cdot p_X(2) + \frac{3}{10} = \frac{4}{5} + 2p_X(2).$$

By solving for $p_X(2)$, we obtain $p_X(2) = 0$ and, consequently, $p_X(0) = 1 - p_X(1) - p_X(2) - p_X(3) = 2/5$. Therefore,

$$E(X^2) = \sum_{x \in \text{supp}(X)} x^2 p_X(x) = 0^2 \cdot p_X(0) + 1^2 \cdot p_X(1) + 2^2 p_X(2) + 3^2 p_X(3)$$

$$= 1 \cdot \frac{1}{2} + 9 \cdot \frac{1}{10} = \frac{14}{10},$$

hence

$$V(X) = E(X^2) - [E(X)]^2 = \frac{14}{10} - \frac{64}{100} = \frac{76}{100} = \frac{19}{25}. $$
4. (8 points) Let $A, B \subset \Omega$ be such that $P(A) = 0.4$, $P(B|A^c) = 0.1$, and $P(B^c|A) = 0.8$. Compute $P(B)$.

**Solution:** We have $B = (A \cap B) \cup (A^c \cap B)$ and the union is a union of disjoint events. Thus, 

$$
P(B) = P(A \cap B) + P(A^c \cap B) = P(A)P(B|A) + P(A^c)P(B|A^c) = P(A)[1 - P(B^c|A)] + [1 - P(A)]P(B|A^c) = 0.4 \times (1 - 0.8) + (1 - 0.4) \times 0.1 = 0.4 \times 0.2 + 0.6 \times 0.1 = 0.08 + 0.06 = 0.14.
$$
5. (10 points) Consider an urn with 3 red balls, 2 blue ball, and 1 white ball. You draw 2 balls from the urn without replacement.

- (1 point) Write the sample space Ω for this experiment (assume that order matters here).
- (1 point) Are the simple events in Ω all equally likely? Clearly justify your answer.
- (2 points) Consider the event $A =$’none of the drawn balls is blue’. List all the simple events of Ω that are contained in $A$
- (6 points) Let $B$ be the event $B =$’at least 1 red ball is drawn’. Compute $P(B)$.


- $Ω = \{RR, RB, RW, BR, BB, BW, WR, WB\}$.
- No, for instance $P(RR) = 3/6 * 2/5 = 1/5$ and $P(RW) = 3/6 * 1/5 = 1/10$.
- $A = \{RR, RW, WR\}$.
- Let $X$ be the random variable counting the number of red balls. Then $X \sim \text{HG} \left(r = 3, N = 6, n = 2 \right)$. We have

$$P(X \geq 1) = 1 - P(X = 0) = 1 - \binom{3}{0} \binom{3}{2} = 1 - \frac{3}{6!} = 1 - \frac{3}{15} = \frac{4}{5}.$$
6. (12 points) Let $X_1, \ldots, X_n \sim \mathcal{N}(\mu, \sigma^2)$. Let $m \in \{0, 1, \ldots, n\}$. What is the probability that exactly $m$ of the $X$’s are larger than 1? Your answer must be expressed as a function of $\Phi$, where $\Phi$ is the c.d.f. of the standard Normal distribution.

Solution: We have

$$P(X_1 > 1) = 1 - P(X_1 \leq 1) = 1 - P\left(\frac{X_1 - \mu}{\sigma} \leq \frac{1 - \mu}{\sigma}\right) = 1 - \Phi\left(\frac{1 - \mu}{\sigma}\right).$$

Because the $X$’s all have the same distribution $P(X_1 > 1) = \cdots = P(X_n > 1) = 1 - \Phi\left(\frac{1 - \mu}{\sigma}\right) = p$. Since the $X$’s are also independent, the number of $X$’s that are larger than 1 is a random variable that has a Binomial distribution with parameters $n$ and $p$. It follows that the probability that exactly $m$ of the $X$’s are larger than 1 is

$$\binom{n}{m} p^m (1 - p)^{n-m} = \binom{n}{m} \left[1 - \Phi\left(\frac{1 - \mu}{\sigma}\right)\right]^m \left[\Phi\left(\frac{1 - \mu}{\sigma}\right)\right]^{n-m}. $$
7. (10 points) Consider the random variable \( X \sim f_X \) with p.d.f.

\[
f_X(x) = k(1 - x)x^31_{[0,1]}(x)
\]

for some \( k > 0 \).

- (3 points) State the name of the distribution of \( X \) and its parameters.
- (3 points) What is the value of \( k \)?
- (2 points) What is \( E(X) \)?
- (2 points) What is \( V(X) \)?

Solution:

- \( X \sim \text{Beta}(\alpha = 4, \beta = 2) \).
- \( k = \Gamma(\alpha + \beta)/[\Gamma(\alpha)\Gamma(\beta)] = \Gamma(6)/[\Gamma(4)\Gamma(2)] = 5!(3!) = 20 \).
- \( E(X) = \frac{\alpha}{\alpha + \beta} = \frac{4}{4 + 2} = \frac{2}{3} \).
- \( V(X) = \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)} = \frac{4\times 2}{(4 + 2)^2(4 + 2 + 1)} = \frac{8}{36 \times 7} = \frac{2}{2 \times 7} = \frac{2}{63} \).
8. (12 points) Let $X \sim \text{Exponential}(\beta)$ and let $\alpha > 0$.

- (8 points) Derive a closed-form expression for $E(X^\alpha)$.
- (4 points) Use the expression that you derived to show that $V(X) = \beta^2$.

**Solution:** Here we exploit what we know about Gamma-type integrals. We have

\[
E(X^\alpha) = \int_0^\infty x^\alpha f_X(x) \, dx = \int_0^\infty x^\alpha \frac{1}{\beta} e^{-\frac{x}{\beta}} \, dx
\]

\[
= \frac{1}{\beta} \int_0^\infty x^\alpha e^{-\frac{x}{\beta}} \, dx = \frac{1}{\beta} \beta^{\alpha+1} \Gamma(\alpha + 1)
\]

\[
= \alpha \Gamma(\alpha) \beta^\alpha.
\]

Then,

\[
V(X) = E(X^2) - [E(X)]^2 = 2 \Gamma(2)\beta^2 - [1 \Gamma(1)]^2 = 2\beta^2 - \beta^2 = \beta^2.
\]
9. (14 points) Consider the pair of random variables \((X_1, X_2) \sim p_{X_1, X_2}\) with joint p.m.f.

\[
p_{X_1, X_2}(x_1, x_2) = \begin{cases} 
0.2 & \text{if } (x_1, x_2) = (0, 0) \\
0.1 & \text{if } (x_1, x_2) = (1, 0) \\
0.1 & \text{if } (x_1, x_2) = (2, 0) \\
0.1 & \text{if } (x_1, x_2) = (0, 1) \\
0.3 & \text{if } (x_1, x_2) = (1, 1) \\
0 & \text{if } (x_1, x_2) = (2, 1) \\
0.05 & \text{if } (x_1, x_2) = (0, 2) \\
0.05 & \text{if } (x_1, x_2) = (1, 2) \\
0.1 & \text{if } (x_1, x_2) = (2, 2). 
\end{cases}
\]

- (4 points) Compute the marginal p.m.f. of \(X_2\).
- (4 points) Compute the marginal c.d.f. of \(X_2\).
- (4 points) Compute the conditional p.m.f. of \(X_1\) given \(X_2 = 1\).
- (2 points) Are \(X_1\) and \(X_2\) independent?

**Solution:**

- Clearly, \(\text{supp}(X_2) = \{0, 1, 2\}\). We have

\[
p_{X_2}(x_2) = \begin{cases} 
p_{X_1, X_2}(0, x_2) + p_{X_1, X_2}(1, x_2) + p_{X_1, X_2}(2, x_2) & \text{if } x_2 \in \{0, 1, 2\} \\
0 & \text{if } x_2 \notin \{0, 1, 2\} 
\end{cases}
\]

\[
= \begin{cases} 
p_{X_1, X_2}(0, 0) + p_{X_1, X_2}(1, 0) + p_{X_1, X_2}(2, 0) & \text{if } x_2 = 0 \\
p_{X_1, X_2}(0, 1) + p_{X_1, X_2}(1, 1) + p_{X_1, X_2}(2, 1) & \text{if } x_2 = 1 \\
p_{X_1, X_2}(0, 2) + p_{X_1, X_2}(1, 2) + p_{X_1, X_2}(2, 2) & \text{if } x_2 = 2 \\
0 & \text{if } x_2 \notin \{0, 1, 2\} 
\end{cases}
\]

\[
= \begin{cases} 
0.2 + 0.1 + 0.1 & \text{if } x_2 = 0 \\
0.1 + 0.3 + 0 & \text{if } x_2 = 1 \\
0.05 + 0.05 + 0.1 & \text{if } x_2 = 2 \\
0 & \text{if } x_2 \notin \{0, 1, 2\} 
\end{cases}
\]

\[
= \begin{cases} 
0.4 & \text{if } x_2 = 0 \\
0.4 & \text{if } x_2 = 1 \\
0.2 & \text{if } x_2 = 2 \\
0 & \text{if } x_2 \notin \{0, 1, 2\} 
\end{cases}
\]
• We have

\[ F_{X_2}(x_2) = P(X_2 \leq x_2) = \begin{cases} 
0 & \text{if } x_2 < 0 \\
p_{X_2}(0) & \text{if } 0 \leq x_2 < 1 \\
p_{X_2}(0) + p_{X_2}(1) & \text{if } 1 \leq x_2 < 2 \\
p_{X_2}(0) + p_{X_2}(1) + p_{X_2}(2) & \text{if } x_2 \geq 2
\end{cases} \]

\[ = \begin{cases} 
0 & \text{if } x_2 < 0 \\
0.4 & \text{if } 0 \leq x_2 < 1 \\
0.4 + 0.4 & \text{if } 1 \leq x_2 < 2 \\
0.4 + 0.4 + 0.2 & \text{if } x_2 \geq 2
\end{cases} \]

• Once again, notice first the conditional distribution is well-defined since \( x_2 = 1 \in \text{supp}(X_2) \) and \( \text{supp}(X_1) = \{0, 1, 2\} \). Then,

\[ p_{X_1|X_2=1}(x_1) = \begin{cases} 
p_{X_1,X_2}(x_1,x_2) & \text{if } x_1 \in \{0, 1, 2\} \\
0 & \text{if } x_1 \notin \{0, 1, 2\}
\end{cases} \]

\[ = \begin{cases} 
p_{X_1,X_2}(0,1) & \text{if } x_1 = 0 \\
p_{X_1,X_2}(1,1) & \text{if } x_1 = 1 \\
p_{X_1,X_2}(2,1) & \text{if } x_1 = 2 \\
0 & \text{if } x_1 \notin \{0, 1, 2\}
\end{cases} \]

\[ = \begin{cases} 
0.1 & \text{if } x_1 = 0 \\
0.3 & \text{if } x_1 = 1 \\
0.4 & \text{if } x_1 = 2 \\
0 & \text{if } x_1 \notin \{0, 1, 2\}
\end{cases} \]

\[ = \begin{cases} 
\frac{1}{4} & \text{if } x_1 = 0 \\
\frac{3}{4} & \text{if } x_1 = 1 \\
0 & \text{if } x_1 \notin \{0, 1\}
\end{cases} \]

• One can notice that \( X_1 \) and \( X_2 \) are not independent in several different ways. Possibly, the easiest way is to notice that the support of the distribution \( p_{X_1|X_2=1} \), which is the set \( \{0, 1\} \), is different from the support of the distribution of \( p_{X_1|X_2=0} \) and \( p_{X_1|X_2=2} \), which is \( \{0, 1, 2\} \) in both cases. One can see that \( p_{X_1|X_2=0} \) and \( p_{X_1|X_2=2} \) have full support \( \{0, 1, 2\} \) by observing that \( p_{X_1,X_2}(x_1,0) \) and \( p_{X_1,X_2}(x_1,2) \) are strictly positive for any \( x_1 \in \{0, 1, 2\} \).
10. (12 points) Consider the pair of random variables \((X_1, X_2) \sim f_{X_1, X_2}\) with joint p.d.f.

\[
f_{X_1, X_2}(x_1, x_2) = k x_1^3 x_2^5 \mathbb{1}_{\{x_1 \geq 0 \wedge x_2 \geq 0 \wedge x_1 + x_2 \leq 1\}}(x_1, x_2)
\]

and \(k > 0\).

- (4 points) Are \(X_1\) and \(X_2\) independent?
- (8 points) Compute the value of \(k\).

Solution:

- \(X_1\) and \(X_2\) are not independent: the support of \(f_{X_1, X_2}\) is not rectangular. Equivalently, the indicator function in the joint density cannot be factorized into the product of two indicator functions such that one only depends on \(x_1\) and the other only depends on \(x_2\).
- Since the p.d.f. must integrate to 1, we have

\[
k = \left( \int_{\mathbb{R}} \int_{\mathbb{R}} f_{X_1, X_2}(x_1, x_2) \, dx_1 \, dx_2 \right)^{-1}.
\]

Therefore,

\[
\int_{\mathbb{R}} \int_{\mathbb{R}} f_{X_1, X_2}(x_1, x_2) \, dx_1 \, dx_2 = \int_0^1 x_2^2 \int_0^{1-x_2} x_1^3 \, dx_1 \, dx_2
\]

\[
= \frac{1}{4} \int_0^1 x_2^2 (1 - x_2)^4 \, dx_2 = \frac{\Gamma(3) \Gamma(5)}{4 \, \Gamma(3 + 5)}
\]

\[
= \frac{1 \cdot 2 \cdot 4!}{4} \times \frac{1}{7 \cdot 6 \cdot 5 \cdot 2} = \frac{1}{420}
\]

and \(k = 420\).